Use of Simulation in Structural Reliability

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ABSTRACT

This paper discusses the use of simulation methods in structural reliability. The main objective is to highlight the utility of simulation in reliability problems that are otherwise difficult or practically impossible to solve. To this aim, Monte Carlo simulation is used as a general tool for reliability analysis of structural systems with respect to the collapse. The structural problem is investigated by means of limit analysis. The usefulness and effectiveness of simulation are shown through the application to the reliability analysis of an existing arch bridge.

INTRODUCTION

Reliability-based concepts are nowadays widely accepted in structural design. However, before such concepts can be effectively applied in the context of probability theory, the design problem often needs to be considerably simplified. In fact, in their basic formulation reliability-based procedures require the structural performance to be represented by explicit relationships among load and resistance variables. In many practical problems, however, these relationships may be unknown, or available only in an implicit form. This happens for example when the reliability of structural systems with nonlinear behavior is investigated (Biondini *et al.* 2004). Moreover, since a complete reliability analysis includes both component-level and system-level evaluations, the use of analytical procedures is usually not feasible for structural systems with several components.

Approximated numerical procedures are available for structural reliability problems that are otherwise difficult or practically impossible to solve by analytical methods. These procedures include first-order reliability methods (FORM) and second order reliability methods (SORM). In general, however, Monte Carlo simulation is the numerical method that has wide applicability to problems involving probability, and often may be the only practical means of finding a solution to a complex probabilistic problem (Ang and Tang 2007).

In this paper, some issues related to the use of simulation methods in structural reliability are discussed. The main objective is to highlight the utility of simulation in solving reliability problems that may be not easily affordable by analytical methods or other numerical techniques. The attention is focused more on practical implementation of simulation methods in reliability analysis procedures, rather than on numerical tools, like variance reduction techniques, aimed to optimize the efficiency of simulation. To this purpose, Monte Carlo simulation in its basic form is used as a general tool for reliability analysis of structural systems with respect to the ultimate limit state of collapse. The structural problem is formulated in the context of limit analysis theory and solved at each simulation cycle by means of linear programming (Biondini 2000). Aspects concerning convergence and accuracy of the simulation process are discussed with reference to a simple benchmark for which the analytical solution exists. The usefulness and effectiveness of simulation is finally shown with the application to the reliability analysis of an existing arch bridge.

PROBABILITY OF FAILURE AND RELIABILITY INDEX

The main objective of structural design is to assure an adequate level of safety against the limit state of collapse. From a deterministic point of view, a structure is safe if the value of the applied loads is no larger than the collapse value, or if the scalar multiplier Θ of the live loads at collapse satisfies the condition $\Theta \geq 1$. Because of the uncertainties involved in the problem (i.e. material and geometrical properties, magnitude and distribution of the loads, among others), the collapse multiplier Θ has to be considered as a random variable and a measure of structural safety is realistically possible only in probabilistic terms. In particular, by denoting with θ an outcome of the random variable Θ, the probability of failure P_F can be evaluated by the integration of the probability density function $f_\Theta(\theta)$ within the failure domain $D = \{\ \theta \mid \theta < 1\}$:

$$P_F = P(\Theta < 1) = \int_D f_\Theta(\theta)\,\mathrm{d}\theta \tag{1}$$

However, a more convenient measure of reliability than $(1-P_F)$ is usually represented by the *reliability index* $\beta = -\Phi^{-1}(P_F)$, where $\Phi = \Phi(s)$ is the standard normal cumulative probability function. The β–index represents, in the space of the standard normal variables (zero mean values and unit standard deviations), the shortest distance from the origin to the limit state surface.

RELIABILITY ANALYSIS

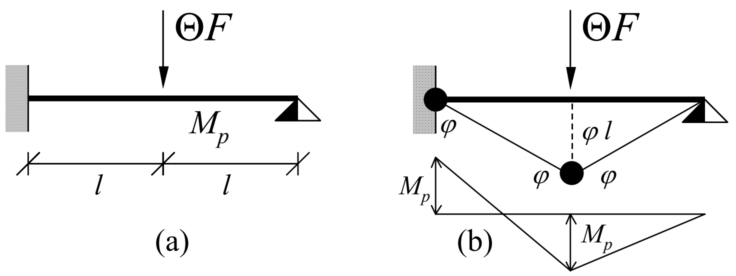
The density function $f_\Theta(\theta)$ depends on a set of random variables $\mathbf{X}=[\ X_1\ X_2\ ...\ X_n\ ]^T$ which define the structural problem (e.g. geometrical and mechanical properties, dead and live loads, among others). This dependency can be expressed in closed form only for very simple problems. Consider for example the beam in Figure 1.a. By assuming a perfectly plastic behavior with resistant bending moment M_p, the limit analysis theory gives the following collapse load multiplier:

$$\Theta = \frac{3M_p}{Fl} \tag{2}$$

with the bending moment distribution and the plastic mechanism shown in Figure 1.b. If F and M_p are statistically independent lognormal random variables, and l is taken as deterministic, it can be shown that the distribution $f_\Theta(\theta)$ of the variable Θ is also lognormal (see Ang and Tang 2007). The statistical parameters of the lognormal random variable Θ can be computed as follows:

$$\mu_\Theta = \frac{3}{l}\frac{\mu_M}{\mu_F}(1+\delta_F^2) \qquad\qquad \delta_\Theta = \sqrt{\delta_F^2 + \delta_M^2 + \delta_F^2\delta_M^2} \tag{3}$$

where μ is the mean value, $\delta = \sigma/\mu$ is the coefficient of variation, σ is the standard deviation, and



	F	M_p
μ	0.50 MN	1.40 MNm
σ	0.15 MN	0.21 MNm
δ	0.30	0.15

(c)

FIGURE 1 – COLLAPSE ANALYSIS OF A BEAM. (A) STRUCTURAL MODEL. (B) BENDING MOMENT DIAGRAM AT COLLAPSE AND COLLAPSE MECHANISM. (C) STATISTICAL PARAMETERS.

the subscripts refer to the random variables F, M_p, and Θ. Based on the lognormal density distribution $f_\Theta(\theta)$, the reliability index can also be directly evaluated as follows:

$$\beta = \frac{\ln\left(\frac{3}{l}\frac{\mu_M}{\mu_F}\right)\sqrt{\frac{1+\delta_F^2}{1+\delta_M^2}}}{\sqrt{\ln[(1+\delta_F^2)(1+\delta_M^2)]}} = \frac{\lambda_\Theta}{\zeta_\Theta} \qquad \lambda_\Theta = \ln\mu_\Theta - \frac{1}{2}\zeta_\Theta^2 \qquad \zeta_\Theta^2 = \ln(1+\delta_\Theta^2) \tag{4}$$

where λ_Θ and ζ_Θ are, respectively, the mean and standard deviation of the normal random variable $\ln\Theta$. With reference to the statistical parameters given in Figure 1.c, and by assuming l=3.00 m, the results listed in Table 1 are finally obtained. It is worth noting that the nominal value of the collapse multiplier associated with the mean values of the variables is $\Theta_{nom}=2.80<\mu_\Theta$.

μ_Θ	σ_Θ	δ_Θ	P_F	β
3.05	1.03	0.338	6.3233×10^{-4}	3.22

TABLE 1 – STATISTICAL PARAMETERS OF THE COLLAPSE LOAD MULTIPLIER Θ, PROBABILITY OF FAILURE P_F, AND RELIABILITY INDEX β FOR THE BEAM SHOWN IN FIGURE 1.

In the presented example the reliability analysis is straightforward. However, if F and M_p are not statistically independent lognormal random variables, an explicit evaluation of the reliability index may be difficult to obtain. More in general, in practical problems the probability density function $f_\Theta(\theta)$ is usually not known, and even if this information is available the evaluation of the probability of failure and the corresponding reliability index can be very difficult. For specialized applications, one possible approach is to use approximated procedures like first-order reliability methods (FORM) and second order reliability methods (SORM). In general, however, a numerical solution is required and Monte Carlo simulation is the numerical process that has wide applicability to problems involving probability (see Ang and Tang 2007).

MONTE CARLO SIMULATION

In Monte Carlo simulation repeated analyses are carried out with random outcomes of the basic random variables $\mathbf{X}=[\, X_1\ X_2\ ...\ X_n\,]^T$ generated in accordance with their i=1,2,…,n marginal density functions $f_{Xi}(x_i)$, or the corresponding cumulative functions $F_{Xi}(x_i)$, as shown in Figure 2. Several procedures are available to perform this generation. For example, it can be shown that an

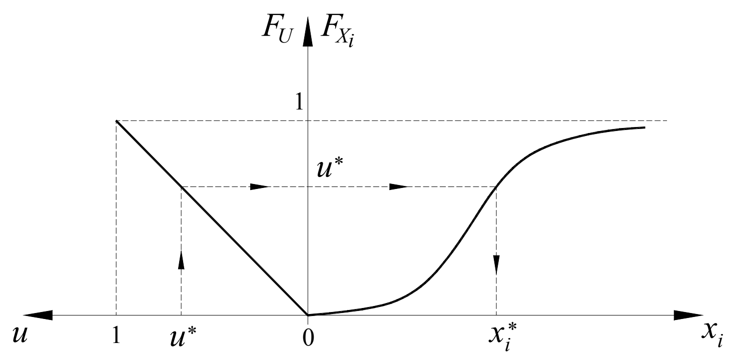


FIGURE 2 – SIMULATION OF A RANDOM OUTCOME x_i OF THE RANDOM VARIABLE X_i ACCORDING WITH THE CUMULATIVE PROBABILITY DISTRIBUTION $F=F_{Xi}(x_i)$.

outcome x of a random variable X normally distributed with mean μ_X and standard deviation σ_X, can be obtained by using the so-called polar method (see Ross 1997). The algorithm is as follows:

Step 1: Generate two random numbers $u_1 \in [0;1]$ and $u_2 \in [0;1]$.

Step 2: Set $v_1=(2u_1-1)$, $v_2=(2u_2-1)$, and $w=(v_1^2 + v_2^2)$.

Step 3: If $w>1$ return to Step 1.

Step 4: Return the unit normal $s = v_1\sqrt{\frac{-2\log w}{w}}$, or $s = v_2\sqrt{\frac{-2\log w}{w}}$.

Step 5: Find out the outcome $x = \mu_X + \sigma_X s$.

If X is instead lognormally distributed, Step 5 should be replaced as follows:

Step 5*a*: Set $\delta_X = \frac{\sigma_X}{\mu_X}$, $\zeta_X^2 = \ln(1+\delta_X^2)$, and $\lambda_X = \ln \mu_X - \frac{1}{2}\zeta_X^2$.

Step 5*b*: Find out the outcome $y = \ln x = \lambda_X + \zeta_X s$, or $x = e^y$.

For Step 1 recursive formulas for generating pseudorandom numbers with uniform distribution are available (see Ross 1997). Moreover, most programming languages have a built-in random number generator. It is worth noting that correlated random variables need to be transformed in uncorrelated variables before the algorithm above can be applied (see Haldar and Mahadevan 2000).

With reference to the problem described in Figure 1, the simulation process can be performed by generating random outcomes of the lognormal random variables F and M_p, and by evaluating the corresponding outcomes of the random variable Θ. This sample of θ–outcomes can be used to build histograms approximating the probability density function $f_\Theta(\theta)$. Figure 3 shows the histograms obtained at different steps N of the simulation process. These histograms are also compared with the lognormal exact solution. The comparison proves the high effectiveness of the simulation process in reproducing the actual distribution of Θ. The rate of convergence of the process is also highlighted in Figure 4.a, which shows the evolution of the computed statistical parameters of the random variable Θ for two different simulations. During the first cycles, the two simulations provide very different results, since each simulation process is different unless the same sequence of random numbers is used. However, similar and quite stable values are reached after about N=500 cycles, and in both cases the error in the estimation of the exact values of the statistical parameters becomes less than 1% after about N=1000 cycles.

The probability of failure P_F and the reliability index β can also be estimated from the results of the simulation process as follows:

$$P_F = \Phi(-\beta) \cong \frac{N_{fail}}{N} \qquad \beta \cong -\Phi^{-1}\left(\frac{N_{fail}}{N}\right) = \Phi^{-1}\left(1-\frac{N_{fail}}{N}\right) \tag{5}$$

where N_{fail} is the number of simulation cycles for which collapse occurred ($\Theta<1$). However, if the distribution type of the random variable Θ is known in advance, or can be properly determined as shown for example in Biondini *et al*. (2004), more accurate estimates of the reliability index can be obtained by using the computed values of the statistical parameters of Θ since in this case, as shown in Figure 4.b, the rate of convergence is expected to be considerably higher.

The accuracy of this approach clearly depends on the number of simulation cycles N, and the minimum number of experiments to achieve a certain level of accuracy depends on the number of random variables involved in the problem, as well as on the value of unknown probability of failure (see Ang and Tang 2007). Since in structural engineering problems many variables are

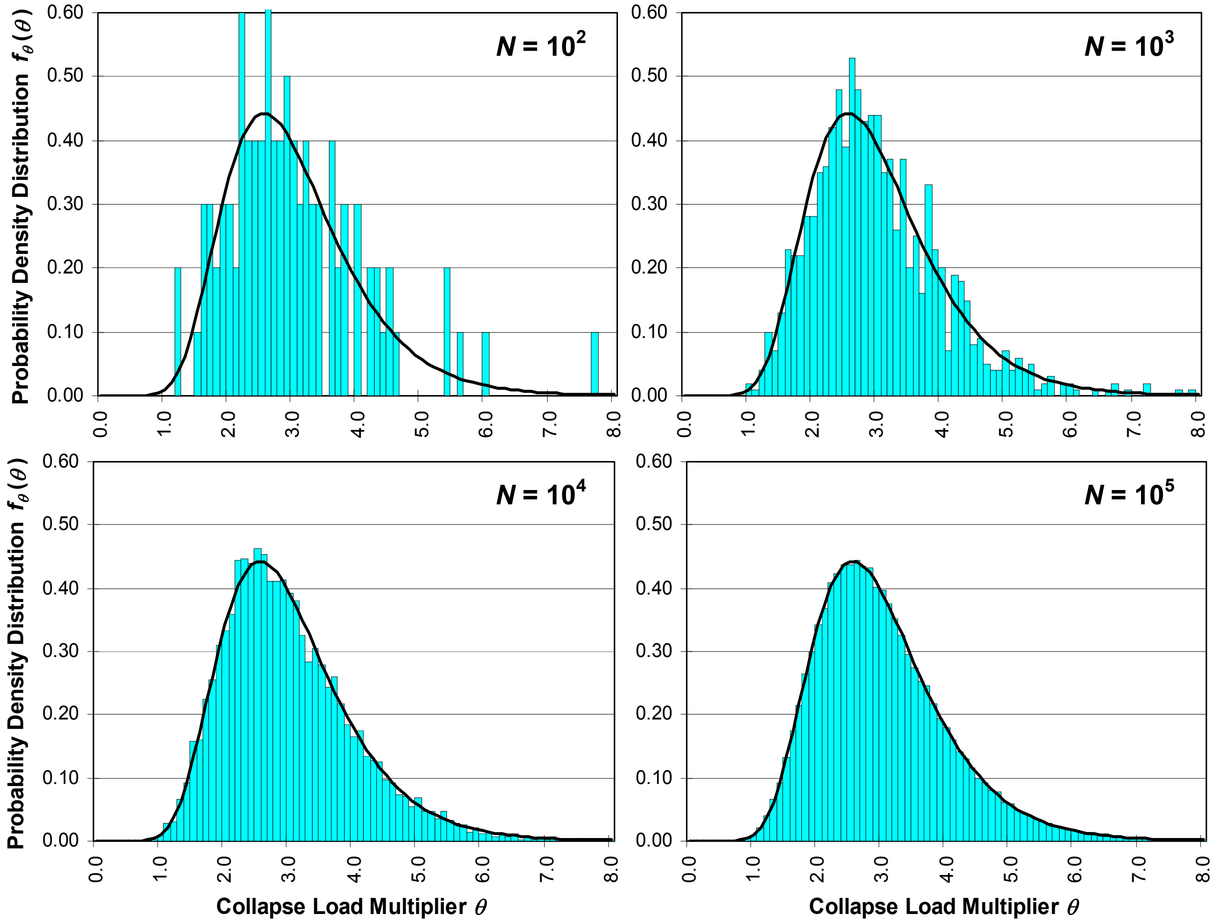


FIGURE 3 – HISTOGRAMS OF THE PROBABILITY DENSITY DISTRIBUTION $f_\Theta(\theta)$ AT DIFFERENT STEPS OF A MONTE CARLO SIMULATION PROCESS FOR THE BEAM SHOWN IN FIGURE 1. THE HISTOGRAMS ARE COMPARED WITH THE LOGNORMAL EXACT SOLUTION OF THE PROBLEM.

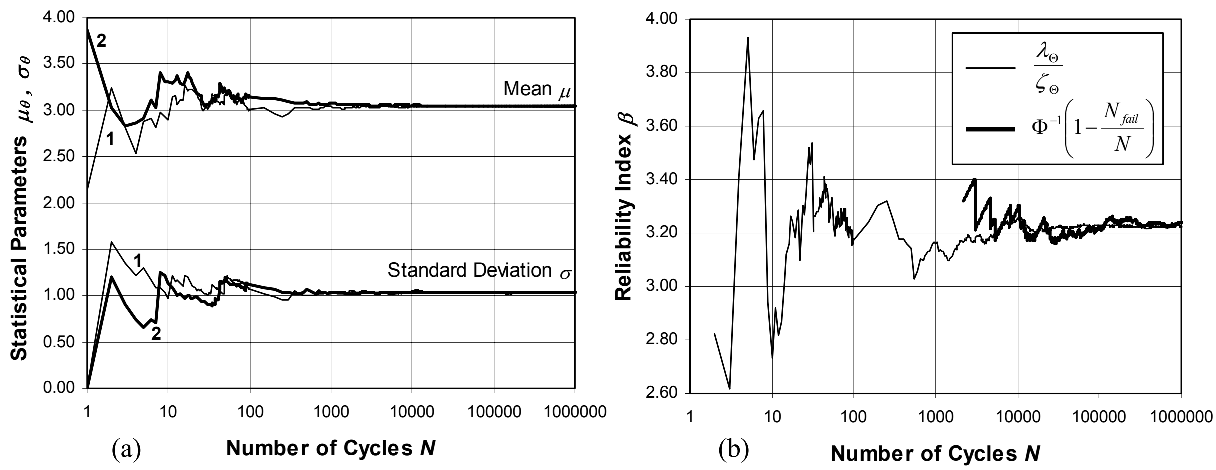


FIGURE 4 – EVOLUTION OF THE SIMULATION PROCESS FOR THE BEAM SHOWN IN FIGURE 1. (A) STATISTICAL PARAMETERS OF THE COLLAPSE LOAD MULTIPLIER Θ (MEAN μ_Θ AND STANDARD DEVIATION σ_Θ) FOR TWO SIMULATIONS. (B) RELIABILITY INDEX FOR SIMULATION 1.

often involved and very small values of probability of failure are accepted ($10^{-9} \div 10^{-5}$), in practice the number of required simulation cycles can be considerably high. With advancements in computer technology the time required to perform a very large number of simulations may become a problem only if the deterministic analysis for each simulation is computationally intensive. In any case, several variance reduction techniques are available to reduce the sampling error and, consequently, the number of cycles required to obtain a given accuracy (see Ross 1997).

RELIABILITY ANALYSIS OF AN ARCH BRIDGE

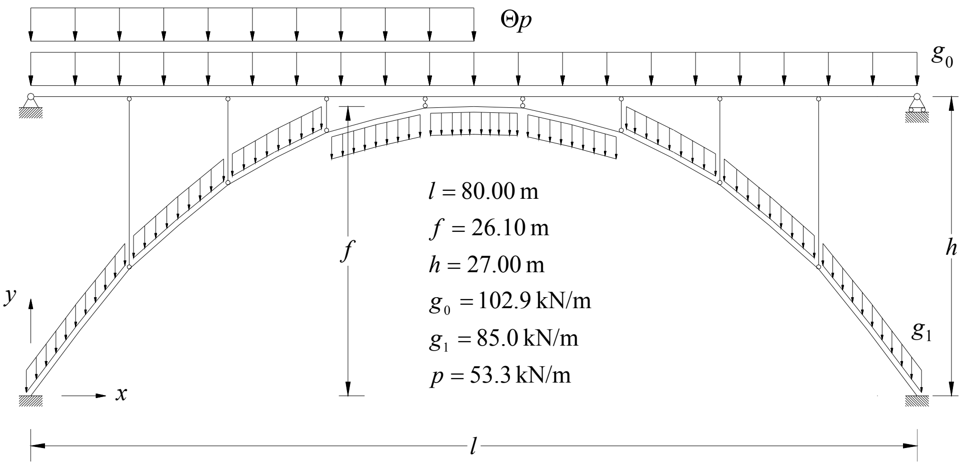
The structural reliability of the existing reinforced arch bridge over the Corace river in Italy is investigated. To this aim, a general approach to limit analysis considering axial force and bending moment as active and interacting generalized plastic stresses is applied (Biondini 2000). In this procedure, based on a stepwise approximation of the axial-force interaction curves, the complete solution of the limit analysis problem, i.e. the collapse load, a stress distribution at incipient collapse, and a collapse mechanism, is obtained by means of linear programming.

The structural model of the bridge is based on the data presented in Galli and Franciosi (1955). The structural scheme and the overall dimensions of the bridge are shown in Figure 5. The arch has uniform cross-section with the axial force-bending moment interaction curve shown in Figure 6.a. This curve is approximated by a four-side stepwise linearization, for the sake of safety inscribed within the resistant domain, as shown in the same Figure 6.a. For the stiffening girder, axially unloaded, the distribution of the resistant bending moments is shown in Figure 6.b. The five supporting walls, simply compressed, are assumed as not critical with respect to collapse. The structure is subjected to a set of dead loads g and to a live load p, as shown in Figure 5. In the limit analysis these distributed loads are replaced by statically equivalent concentrated loads, 12 for each span of the girder and 6 for each span of the arch. Additional details on this structural model can be found in Biondini and Frangopol (2008), where the life-cycle performance of the arch bridge subject to environmental damage is investigated.

All data given in Figures 5 and 6 are assumed as nominal values. The limit analysis for the nominal scenario provides a collapse multiplier $\Theta_{nom} = 4.28$. Figure 7 shows a possible collapse mechanism, together with a possible distribution of axial force and bending moment at collapse.

The probabilistic analysis is carried out by considering the following random variables: (a) The location (x, y) of the nodal connections of the structural model (i.e. the ends of each arch segment, and the ends of each girder segment having different properties); (b) The distance k of each side of the resistance curve from the origin, for each structural member, in the hypothesis that the shape of the resistance domains is not affected by significant randomness; (c) The magnitude of dead loads g and live loads p acting in each span of both the arch and the girder. The distribution types and statistical parameters assumed for these variables are listed in Table 2. To emphasize the effects of the uncertainties, no correlation is considered among these variables.

Figure 8.a shows the histogram of the probability density function $f_\Theta(\theta)$ obtained after $N = 10^6$ simulation cycles. Based on the results of chi-square and Kolmogorov-Smirnov tests for goodness-of-fit (see Ang and Tang 2007), a lognormal distribution is selected as a very appropriate model for $f_\Theta(\theta)$. This model is included in Figure 8.a, where the goodness-of-fit of the lognormal distribution can be verified visually as well. The rate of convergence of the computed statistical parameters is shown in Figure 8.b. The best estimates obtained for these parameters are μ_Θ=4.04, σ_Θ=0.74, and $\delta_\Theta=\sigma_\Theta/\mu_\Theta$=0.183. A reliability index β=7.62 is obtained from the lognormal model. Since this value corresponds to a probability of failure $P_F \approx 10^{-14}$, at least N=10^{14} simulation cycles would be required, on average, to find an outcome θ<1. As a consequence, in this case



x [m]	y [m]
0.00	0.00
4.44	5.90
8.89	11.43
13.35	15.39
17.80	19.04
22.20	21.53
26.70	23.76
31.20	24.99
35.60	25.98
40.00	26.10

Arch Shape

FIGURE 5 – ARCH BRIDGE. OVERALL DIMENSIONS AND LOADS (NOMINAL SCENARIO).

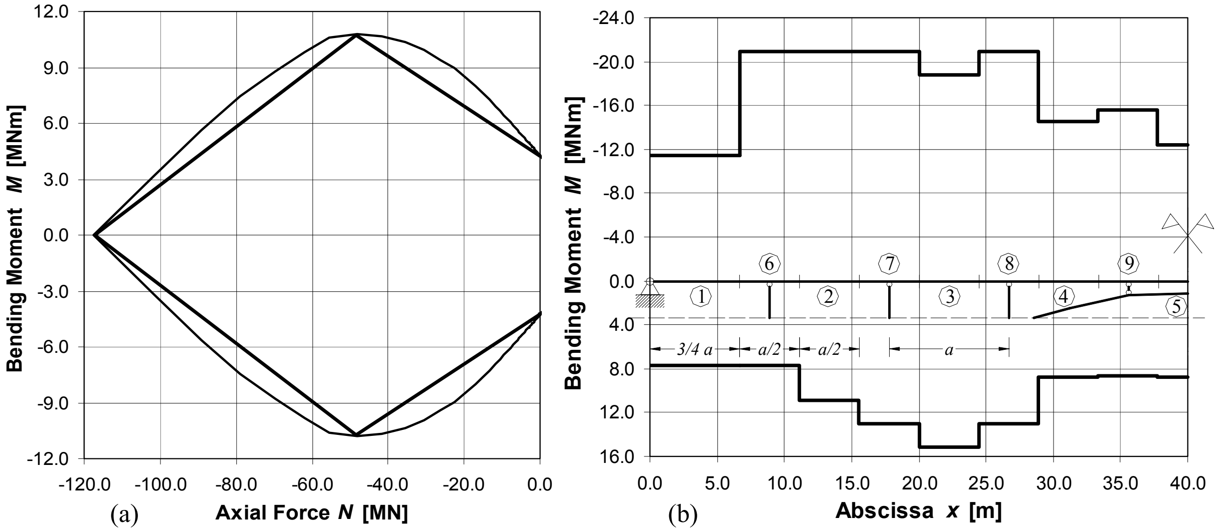


FIGURE 6 – ARCH BRIDGE. (A) AXIAL FORCE-BENDING MOMENT RESISTANT DOMAIN FOR THE ARCH. (B) DISTRIBUTION OF THE RESISTANT BENDING MOMENTS IN THE GIRDER.

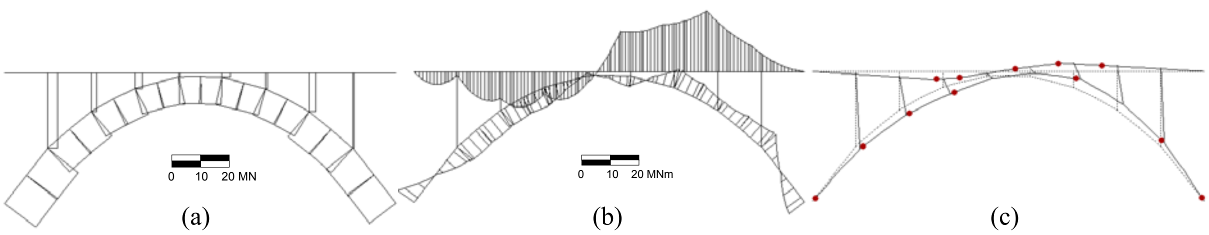


FIGURE 7 – ARCH BRIDGE. RESULTS OF LIMIT ANALYSIS FOR THE NOMINAL SCENARIO (Θ_{nom}=4.28). (A) AXIAL FORCE AND (B) BENDING MOMENT DIAGRAMS AT COLLAPSE. (C) COLLAPSE MECHANISM.

Variable	Distribution Type	μ	σ
(Δx, Δy)	Normal	0	50 mm
k	Lognormal	k_{nom}	0.20 k_{nom}
g	Normal	g_{nom}	0.10 g_{nom}
p	Normal	p_{nom}	0.40 p_{nom}

TABLE 2 – ARCH BRIDGE. DISTRIBUTION TYPES AND STATISTICAL PARAMETERS.

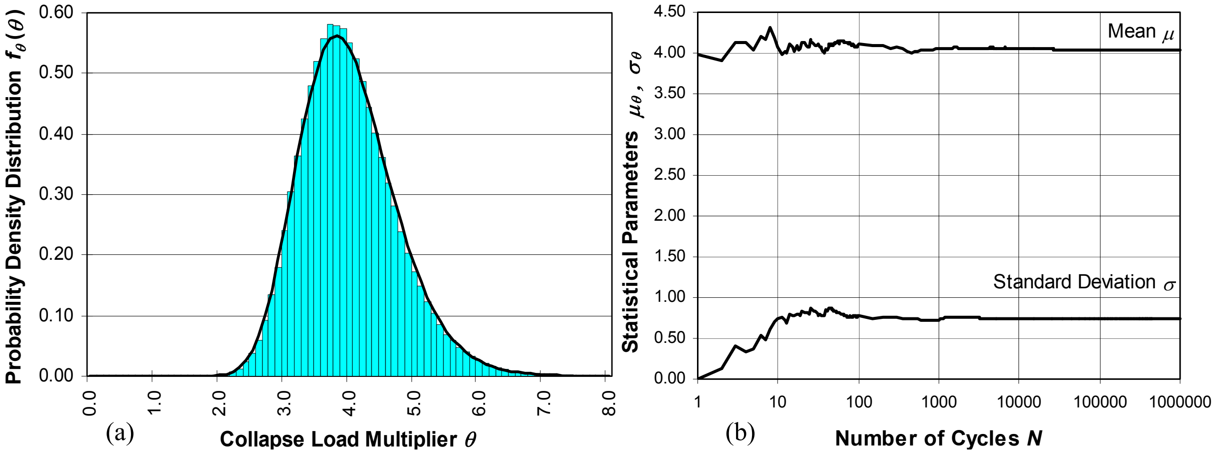


FIGURE 8 – ARCH BRIDGE. RESULTS OF THE SIMULATION PROCESS AFTER N=10^6 CYCLES: (A) HISTOGRAM OF THE DENSITY FUNCTION $f_\Theta(\theta)$ COMPARED WITH A LOGNORMAL DISTRIBUTION MODEL, AND (B) STATISTICAL PARAMETERS OF THE COLLAPSE LOAD MULTIPLIER Θ.

the number N_{fail} cannot be used to obtain reliable estimates of β, unless a very high number of simulation cycles is carried out (e.g. $N > 10^{16} \div 10^{17}$). This confirms the crucial role played by a proper selection of the distribution type for the random variables under investigation. However, when a distribution suitable to fit the sample data is not available, variance reduction techniques can be applied to reduce the number of required simulation cycles.

CONCLUSIONS

The usefulness and effectiveness of simulation methods in reliability problems involving structural systems with several components and implicit relationships among load and resistance variables, have been shown with reference to the probabilistic limit analysis of an existing arch bridge. The results of the simulation process, in terms of reliability estimates, highlighted the crucial role played by a proper selection of the distribution type for the random variables under investigation.

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