PROBABILISTIC LIMIT ANALYSIS OF FRAMED STRUCTURES

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Abstract

The paper deals with the reliability assessment of framed structures under static loading conditions. The safety against the structural collapse should be described with reference to the limit state which separates the stable states of the structure from the unstable ones. This event is investigated by means of limit analysis through a systematic approach which considers axial force and bending moment as active and interacting generalized plastic stresses. However, due to the uncertainties in some material and geometrical properties, in the magnitude and distribution of the loads, etc., the collapse load so obtained must be considered as a random variable and the structural safety assured in probabilistic terms. This problem is solved by a Monte Carlo simulation, where repeated limit analyses are carried out to build a meaningful set of experimental data. Based on the sample so obtained, the collapse load distribution is derived and a reliability index is evaluated. The procedure is finally applied to the reliability analysis of a reinforced concrete arch bridge structure.

Introduction

The safety against the structural collapse should generally be described with reference to the limit state which separates the stable states of the structure from the unstable ones. Since it usually depends on internal quantities of the system in an implicit way, a check of the structural safety can be carried out only after suitable structural models able to describe the fundamental behavior of the structure have been selected. In many cases, this aim can be pursued by assuming perfectly plastic behavior and negligible second order effects, which, as known, make the general theory of the limit analysis applicable. In spite of such idealizations, this theory seems to work very well, at least for the prediction of the failure loads, even in cases where the actual structural behavior is far from the idealized one. This happens also for concrete structures if the tensile strength is neglected and the compression strength is properly modified through a suitable effectiveness factor.
Based on previous considerations, the first part of the paper develops a systematic approach to the limit analysis of plane framed structures, which considers axial force and bending moment as active and interacting generalized plastic stresses (Martinez *et al.* 1997). The structural problem is reduced to algebraic form and the complete solution, i.e. the collapse loads, a stress distribution at the incipient collapse and a collapse mechanism, is obtained by means of linear programming. However, because of the uncertainties in some material and geometrical properties, in the magnitude and distribution of the loads, etc., such solution is affected by randomness. In particular, the failure load must be considered as a random variable and the structural safety can be realistically assured only in probabilistic terms. In this work, the problem is solved by a Monte Carlo simulation, where repeated limit analyses are carried out to build a meaningful sample of experimental data. A final application to a R.C. arch bridge structure shows the effectiveness of the procedure.

Limit Analysis under Combined Plastic Stresses

Equilibrium and Compatibility Conditions

Forces and generalized stresses are assumed in accordance with the conventions and with the reference systems in Figure 1. In the following, *equilibrium* and *compatibility* conditions are derived on the basis of the classical *small displacements hypothesis*.

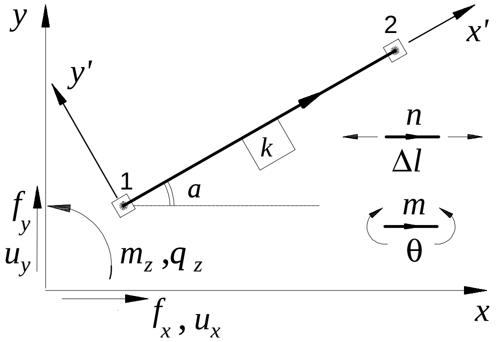

Figure 1. Reference systems.

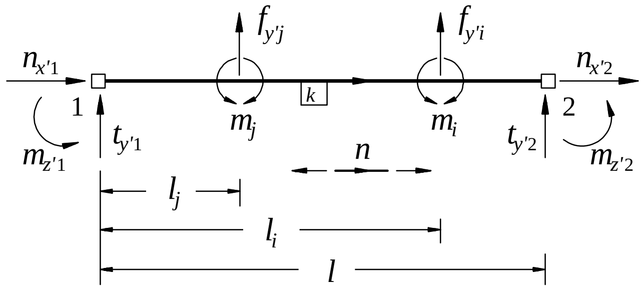

Figure 2. Reference statical quantities.

The end forces and the internal generalized stress of a beam element can be posed as a function of the applied loads and of three independent statical quantities (Livesley 1975). Since it is reasonable to substitute a distributed load with statically equivalent concentrated loads in an appropriate number of sections, the element is considered subjected only to concentrated forces normal to the beam axis. When axial loads or applied moments are present, the element can be further subdivided. In this way, is convenient to select as reference quantities the axial force n and the end moments $\bar{m}_1 = -m_{z'1}$ and $\bar{m}_2 = m_{z'2}$ (Figure 2). Thus, in the local coordinate system the following relationships hold:

$$\mathbf{f}'_1 = \mathbf{H}'_1\mathbf{r} - \mathbf{f}_1^{e'} \qquad \mathbf{f}'_2 = \mathbf{H}'_2\mathbf{r} - \mathbf{f}_2^{e'} \tag{1}$$

where $\mathbf{f}'_1 = [n_{x'1} \;\; t_{y'1} \;\; m_{z'1}]^T$, $\mathbf{f}'_2 = [n_{x'2} \;\; t_{y'2} \;\; m_{z'2}]^T$, $\mathbf{r} = [n \;\; \bar{m}_1 \;\; \bar{m}_2]^T$, while $\mathbf{f}_1^{e'}$, $\mathbf{f}_2^{e'}$, $\mathbf{H}'_1$, $\mathbf{H}'_2$, are obtained from equilibrium conditions (Figure 2). These equations can be rewritten in the global reference system for each node s and assembled on the whole structure:

$$\sum\nolimits_{k \to s} (\mathbf{f}_{hk} + \mathbf{f}_{hk}^{e}) = \sum\nolimits_{k \to s} \mathbf{H}_{hk}\mathbf{r}_k \qquad \Rightarrow \qquad \mathbf{f}_A = \mathbf{H}_A\mathbf{r}_A \tag{2}$$

where the sums regard all the elements k whose end h (h=1,2) converges at the node s. Besides the ends, the more critical sections of the element are those directly loaded. The bending moments in each section i can also be expressed as a function of the applied loads:

$$m_i^0 = \mathbf{h}_i^T\mathbf{r} + m_i \qquad \Rightarrow \qquad \mathbf{f}_B = \mathbf{H}_B\mathbf{r}_A + \mathbf{r}_B \tag{3}$$

where $\mathbf{h}_i$ and m_i^0 may be obtained again by simple equilibrium (Figure 2). In conclusion, the generalized stress vector $\mathbf{r} = [\mathbf{r}_A^T \,\vdots\, \mathbf{r}_B^T]^T$ can be directly correlated to the applied loads vector $\mathbf{f} = [\mathbf{f}_A^T \,\vdots\, \mathbf{f}_B^T]^T$, through an equilibrium matrix $\mathbf{H}$ as follows:

$$\mathbf{f} = \mathbf{H}\mathbf{r} \tag{4}$$

The generalized strains corresponding to the stresses n, $\bar{m}_1$, $\bar{m}_2$, m_i, are the elongation Δl, and the rotations $\bar{q}_1$, $\bar{q}_2$, q_i. Thus, to the stress vector $\mathbf{r}$ is associated the strain vector $\mathbf{e} = [\mathbf{e}_A^T \,\vdots\, \mathbf{e}_B^T]^T$, with $\mathbf{e}_A = [\mathbf{e}_1^T \,\vdots\, \mathbf{e}_2^T \,\vdots\, \ldots]^T$, $\mathbf{e}_k = [\Delta l \;\; \bar{q}_1 \;\; \bar{q}_2]^T$, $\mathbf{e}_B = [q_1 \;\; q_2 \;\; \ldots]^T$. In an analogous way, to the load vector $\mathbf{f}$ corresponds a vector of displacements in the global reference system $\mathbf{s} = [\mathbf{s}_A^T \,\vdots\, \mathbf{s}_B^T]^T$, with $\mathbf{s}_A = [\mathbf{s}_1^T \,\vdots\, \mathbf{s}_2^T \,\vdots\, \ldots]^T$, $\mathbf{s}_k = [u_x \;\; u_y \;\; q_z]^T$, $\mathbf{s}_B = \mathbf{e}_B$. One can verify that between displacements $\mathbf{s}$ and strains $\mathbf{e}$, the following relationship holds:

$$\mathbf{e} = \mathbf{H}^T\mathbf{s} \tag{5}$$

Yield Conditions and Flow Rule

The general theory of limit analysis assumes *rigid perfectly-plastic constitutive laws*. It follows that (*a*) the *yielding criterion*, which defines the stress state corresponding to the starting of the plastic flow, is convex and (*b*) the *flow rule*, through which the increments of the plastic strains are correlated to the stress state, is associated to the yielding surface.

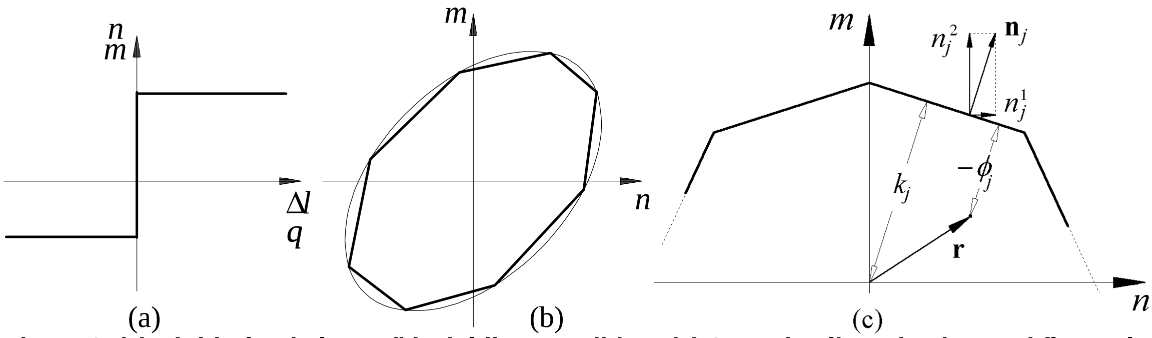

Figure 3. (a) Rigid-plastic laws. (b) Yielding condition. (c) Stepwise linearization and flow rule.

By assuming the normal force n and the bending moment m the only active generalized plastic stresses (Figure 3.a), the yielding criterion for the generic critical section i can be written as $f_i(n_i, m_i) = 0$. Such a criterion defines in the (n-m) plane a curve which can be reasonably idealized by a stepwise approximation, for safety's sake inscribed within the convex domain $f_i(n_i, m_i) \leq 0$ (Figure 3.b). Thus, by assuming a stepwise linearization with q_i sides, the yielding criterion for section i, and then for the whole structure, is rewritten as:

$$\phi_i = \mathbf{N}_i \mathbf{r}_i - \mathbf{k}_i \leq 0 \quad \Rightarrow \quad \phi = \mathbf{N}\mathbf{r} - \mathbf{k} \leq 0 \tag{6}$$

where $\phi_i = [f_1^i \; f_2^i \; \dots \; f_{q_i}^i]^T$, $\mathbf{N}_i = [\mathbf{n}_1^i \; \mathbf{n}_2^i \; \dots \; \mathbf{n}_{q_i}^i]^T$, $\mathbf{n}_j^i = [n_{j1}^i \; n_{j2}^i]^T$, $\mathbf{k}_i = [k_1^i \; k_2^i \; \dots \; k_{q_i}^i]^T \geq \mathbf{0}$, $\mathbf{r}_i = [n_i \quad m_i]^T$ (Figure 3.c). The associated flow rule for section i is given by the equations:

$$\Delta l_i = \mu_i \frac{\partial f_i}{\partial n_i} \qquad q_i = \mu_i \frac{\partial f_i}{\partial m_i} \tag{7}$$

with the multiplier $\mu_i \geq 0$ that allows plastic flows only for the points lying on the yielding curve along the outside normal (Figure 3.c), that is $\mu_i f_i(n_i, m_i) = 0$. For the linearized case:

$$\mathbf{e}_i = \mathbf{N}_i^T \mu_i \quad \Rightarrow \quad \mathbf{e} = \mathbf{N}^T \mu \tag{8}$$

with $f_j^i \mu_j^i = 0$ (j=1,...,q_i) and where $\mathbf{e}_i = [\Delta l_i \; q_i]^T$, $\mu_i = [\mu_1^i \; \mu_2^i \dots \mu_{q_i}^i]^T \geq \mathbf{0}$, $\Delta l = \sum_{i \in k} \Delta l_i$, being the sum extended to all the critical sections i of the element k.

Static and Kinematic Approach (Duality of the Solution)

Let $\mathbf{f}_0$ be a vector of constant loads and $\mathbf{f}$ a vector of loads whose intensity is proportional to a given scalar multiplier $\lambda \geq 0$. By assuming that the structure is stable for λ=0, the multiplier associated to its failure, or the *collapse multiplier* λ_c, can be defined on the basis of the two fundamental theorems of limit analysis. The *lower bound theorem* states that λ_c is the maximum of the multipliers associated to stress fields which satisfy both the equilibrium conditions and the yielding criterion. In a dual way, the *upper bound theorem* states that λ_c is the minimum of the multipliers associated to plastic flows which satisfy both the compatibility conditions and the flow rule. In mathematical terms:

$$\max \left\{ \lambda \;\middle|\; (\lambda \mathbf{f} - \mathbf{H}\mathbf{r}) = -\mathbf{f}_0, \; \mathbf{N}\mathbf{r} \leq \mathbf{k}, \; \lambda \geq 0 \right\} \tag{9}$$

$$\min \left\{ \mathbf{k}^T \mu - \mathbf{f}_0^T \mathbf{s} \;\middle|\; (\mathbf{N}^T \mu - \mathbf{H}^T \mathbf{s}) = \mathbf{0}, \; \mathbf{f}^T \mathbf{s} = 1, \; \mu \geq \mathbf{0} \right\} \tag{10}$$

These linear programs lead to the *complete solution* of the problem, i.e. the collapse multiplier, the stress distribution at the incipient collapse and the collapse mechanism. Being the separator of two sets, λ_c is unique. However, the uniqueness of λ_c doesn't necessarily mean the uniqueness of the collapse mechanism, nor that of the collapse stress field.

Structural Reliability Assessment by Monte Carlo Simulation

In the frame of the limit analysis, a structure is safe if the load multiplier l is no larger than its collapse value l_c. Because of the uncertainties involved in the problem, the quantity l_c has to be considered as a random variable and the assurance of structural safety is realistically possible only in probabilistic terms. In particular, by putting $\bar{\lambda}$ an outcome of the random variable l_c, the probability of failure can be evaluated by the integration of its density function $f_{l_c}(\bar{\lambda})$ within the failure domain $D = \{ \bar{\lambda} \mid \bar{\lambda} \leq l \}$:

$$P_F = P(l_c \leq l) = \int_D f_{l_c}(\bar{\lambda})\, \mathrm{d}\bar{\lambda} = \Phi(-b) \tag{11}$$

being $b = -\Phi^{-1}(P_F)$ a *reliability index* and $\Phi = \Phi(s)$ the standard normal cumulative function. In practice the density function $f_{l_c}(\bar{\lambda})$ is not known and at the most some information is available only about a set of basic random variables **X** which define the structural problem (e.g. geometrical **H** and mechanical properties **N** and **k**; dead $\mathbf{f}_0$ and live loads **f**). Since the outcomes of l_c are dependant on the outcomes of **X**, i.e. $l_c = l_c(\mathbf{X})$, if a failure condition is introduced as $g(\mathbf{X}) \leq 0$, the associated probability may be evaluated by the integration of the assumed known joint density function $f_{\mathbf{X}}(\mathbf{x})$ within the failure domain $D = \{ \mathbf{x} \mid g(\mathbf{x}) \leq 0 \}$:

$$P_F = P[g(\mathbf{X}) \leq 0] = \int_D f_{\mathbf{X}}(\mathbf{x})\, \mathrm{d}\mathbf{x} = \Phi(-b) \tag{12}$$

This formulation leads to a stochastic programming problem (Gavarini 1969), which usually result very expensive to solve. A more effective approach refers to simulation methods, which usually give very good results (Casciati and Sacchi 1974). In this work, the problem is solved by using a Monte Carlo simulation where repeated analyses are carried out with random outcomes of the basic variables **X** generated in accordance to the function $f_{\mathbf{X}}(\mathbf{x})$ (Bontempi *et al.* 1998). Based on the sample so obtained, the density function $f_{l_c}(\bar{\lambda})$ for the limit state $l_c = l$ is derived and the reliability index $b = b(l)$ is evaluated.
The accuracy of this approach clearly depends on the number N of experiments. Several techniques are available either to reduce the sampling error, or to estimate the required sample size to obtain a given accuracy. In this work, the sampling error is reduced by the antithetic variables technique, while a posteriori estimation on the goodness of the sample size is based on a monitoring of the function $b = b(N)$ for each multiplier l of interest.

Application to a R.C. Arch Bridge Structure

The R.C. arch bridge structure shown in Figure 4 is considered (Ronca and Cohn 1979). The arch has three different cross sections for which the yielding curves $f(n,m)=0$ are idealized by a four-sides stepwise linearization (Figure 5). For the stiffening girder, being axially unloaded, the bending moment is considered the only generalized plastic stress, with yield limits $m = \pm m_p$. The five supporting walls, simply compressed, are instead assumed as not critical with respect to the collapse. The structure is subjected to a set of dead loads g and to a live load p (Figure 4). These distributed loads are replaced by statically equivalent concentrated loads, 9 for each span of the girder and 3 for each span of the arch. Geometrical dimensions, mechanical strengths and load intensities given in Figures 4 and 5, are assumed as nominal values. The limit analysis of this *nominal structure* leads to the collapse multiplier $l_c = 3.995$. Figure 6 shows a possible collapse mechanism, while a set of stress values in the corresponding generalized plastic hinges which develop in the arch are marked on the interaction curves in Figure 5. The reliability of the structure is analyzed by assuming as random variables: (a) the location (x, y) of the nodal connections between the structural elements; (b) the size k of the no yielding regions for each structural member, in the reasonable hypothesis that their shape is not affected by significant randomness; (c) the loads g and p in each span. All these variables are considered statistically independent and distributed with mean and standard deviation values as specified in Table 1.

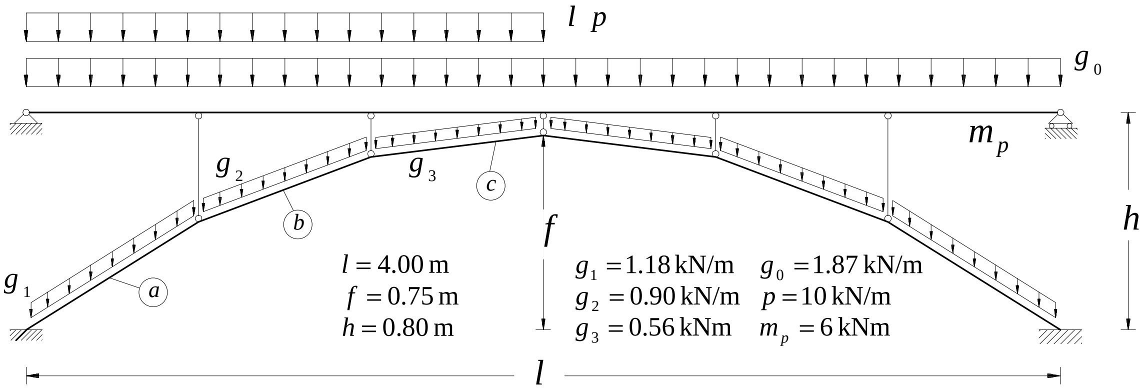

Figure 4. Arch bridge (nominal) structure. Overall dimensions and load condition.

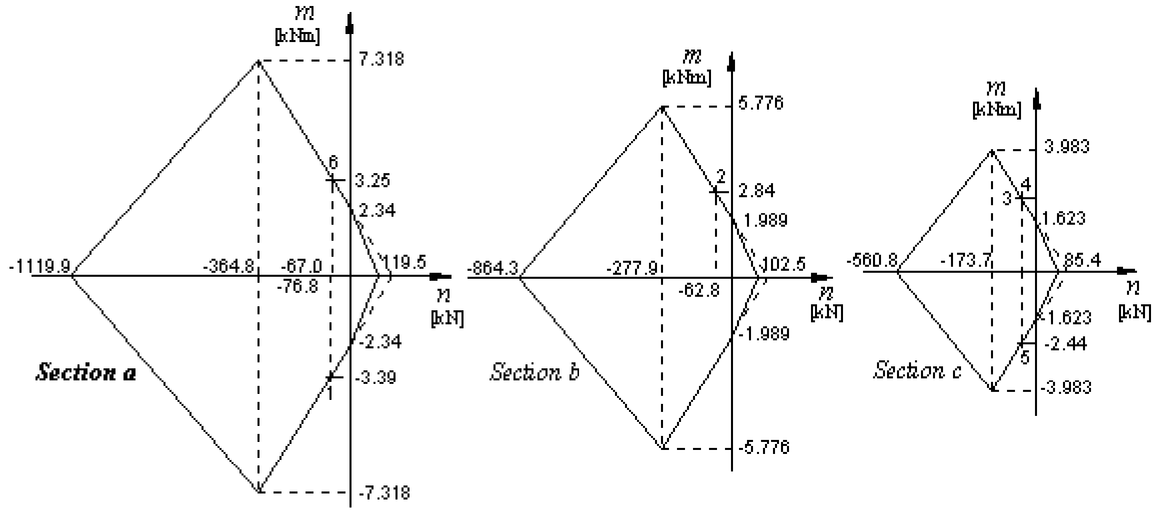

Figure 5. Yielding curves for the arch cross sections *a*, *b*, and *c* (see Figure 4) and a set of stress values in the generalized plastic hinges (see Figure 6) for the nominal structure.

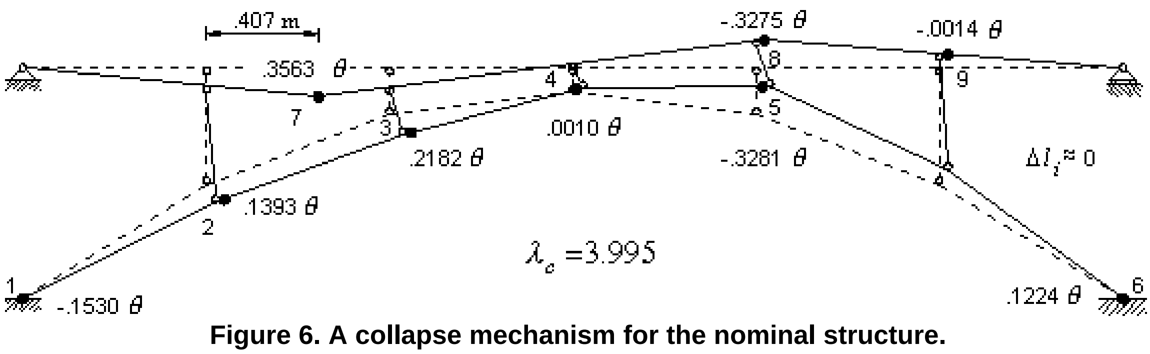

Figure 6. A collapse mechanism for the nominal structure.

Variable	Type	m	s	Affected Quantities
(x, y)	N	$(x, y)_{\text{nom}}$	50 mm	$\mathbf{H}$, $\mathbf{f}_0$, $\mathbf{f}$
k	LN	k_{nom}	$0.20\,k_{\text{nom}}$	$\mathbf{k}$
g	N	g_{nom}	$0.10\,g_{\text{nom}}$	$\mathbf{f}_0$
p	N	p_{nom}	$0.40\,p_{\text{nom}}$	$\mathbf{f}$

Table 1. Random variables (N=Normal; L=Lognormal; m=mean; s=standard deviation).

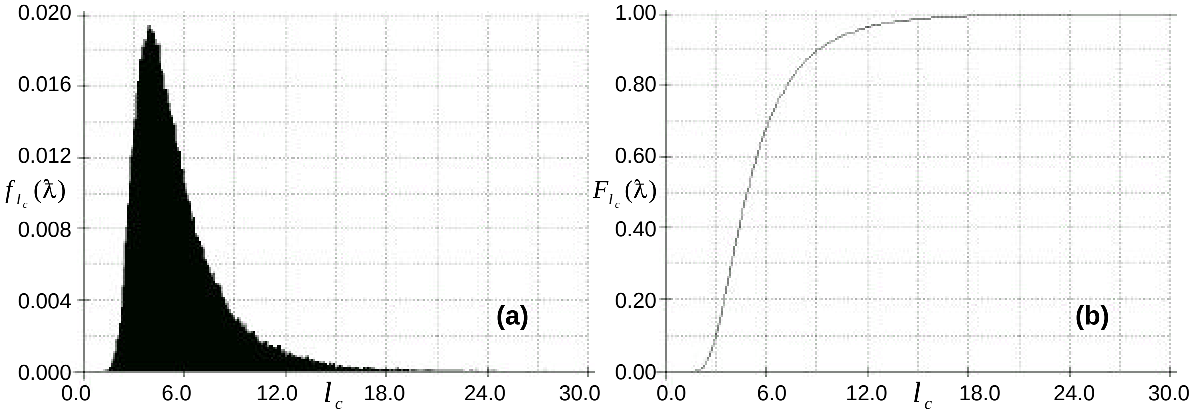

Figure 7. (a) Density $f_{l_c}(\lambda)$ and (b) cumulative function $F_{l_c}(\lambda) \equiv P_F$ after *N*=10^5 simulations.

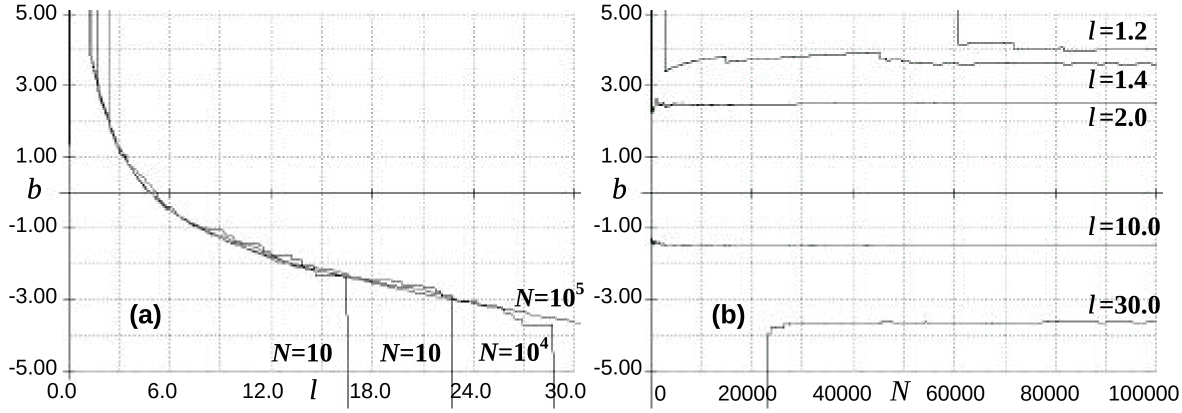

Figure 8. Reliability index *b*=*b*(*l*,*N*). (a) *b*=*b*(*l*) after *N* simulations. (b) *b*=*b*(*N*) for a given *l*.

After *N*=100000 experiments, the simulation leads to the density function in Figure 7.a and to its cumulative in Figure 7.b. The structural reliability for a given multiplier *l* is then fully described by this probability of failure, or by the distribution of the corresponding reliability index *b*. The diagrams of both the functions *b*=*b*(*l*) for an increasing number *N* of simulations and *b*=*b*(*N*) for several values of *l* are shown in Figure 8.a and 8.b, respectively. The inspection of such diagrams leads to appreciate how the sample size *N* needful to reach a stable *b*–index depends on the load multiplier *l*, and progressively increases as the limit state of collapse moves towards the tails of the probability distributions.

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