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# Optimal strut-and-tie models in reinforced concrete structures

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A general solution of the diffusion problems which concern the R.C. structures, may be deduced by the limit analysis, by means of truss schemes suitable to model the load transfer mechanism. In particular such schemes allow us to share the carrying functions between concrete and steel reinforcement. Latest developments call this kind a solution Strut-and-Tie (S&T) modellization. In this paper a procedure for the automatic search for optimal S&T models in R.C. elements is proposed. A highly indeterminate pin-jointed framework of a given layout is generated within the assigned geometry of the concrete element and an optimum truss is found by the minimization of a suitable objective function. Such a function allows us to search for the optimum truss according to a reference behaviour (the principal stress field) deduced through a F.E.A. and assumed as representative of the given continuum.

After having explained the theoretical principles and the mathematical formulation, some examples show the pratical application of the procedure and its capability in handling complex stress paths, through schemes which result rational and suitable for a consistent design.

## 1. Introduction

The design of R.C. elements can be viewed as the translation of the results of the structural analysis onto a reinforcement layout, which according to the resistant capacities of the steel and of the concrete, shares their carrying role in the composite continuum. Well defined criteria for the cases of axial flexure, shear and torsion states exist in the design of slender beams. Such criteria assume certain kinematic hypotheses on the strain state (*e.g.* the sections rotate remaining plane) and allow us to deduce, from a conventional state represented by the generalized sectional forces ($N$, $T_y$, $T_z$, $M_y$, $M_z$, $M_t$), reinforcement assemblies which work according to a 2D, 3D carrying mechanism, as is assumed for the shear and torsion cases. When a structural element can no longer be viewed as a De Saint Venant beam, a sufficiently simple deformative model is not available to solve the problem with similar criteria. In these cases the design procedures refer to experiments and schemes specialized in characteristic situations (deep beams, dapped end beams, corbels, *etc.*).

From the point of view of the structural analysis, the study of any complex structure can be carried out by means of the F.E.M. The design of the same structure as a R.C. element is neither immediate nor unique. For instance, one can subdivide the flow of the tensile stresses into influence areas and, working at an assigned level of stress, attribute a corresponding amount of reinforcement to each of them. This, in truth, is a debatable criteria. In this way, for instance, the bars of reinforcement in a rectangular simply supported beam, will result proportionally distributed along the entire depth in tension, instead of near the bottom edge, as the R.C. theory suggests. Moreover similar *automatic* criteria consider a single component of stress, generally along one of the principal directions, and even if they allow us an estimate of the reinforcement amount, they tend to neglect any check on the concrete which in absence of suitable thickness and/or fitting local reinforcement details, may be prone to early failures. Hence one can say that there isn't a general procedure for passing from a given stress field to a corresponding resistant scheme.

In the theory of R.C., schemes of resistant mechanisms having some similarity to the methods of the limit analysis, have been employed since the beginning of the century. They were not immediately related to the theory of plasticity and they were used to model the shearing and torsional behaviour of slender beams [20, 21, 23]. The classical truss model was then applied to study many cases of diffusion regions. The initial hypotheses were widely corroborated by experimental studies, through which many aspects which were difficult to deduce a priori were defined (for instance the collaborating areas of the compressed struts). Many Conferences and Colloquia contributed in defining the actual state of art [11, 12, 13]. At present the Strut-and-Tie (S&T) modellization is proposed as a common tool for a general and *consistent* design of R.C. structure [1, 14, 18, 19, 26, 27].

A S&T model is generally formed by concrete struts, steel ties and nodal zones, intended as the polygonal areas surrounding the intersection among the axes of the bars and/or of the lines along which the loads and the reactions act. If the equilibrium and the limiting strength of the materials are satisfied, such a model, according to the lower bound theorem of the theory of plasticity, leads us to underestimate the ultimate load carrying capacity of the structure [7].

The usual criteria configure a S&T model which moves from a reference stress field and disposes the elements of the truss following the path of the isostatics by modelling their curvilinear flow through polygonal lines. The multiaxial stress states inside the nodal regions, generally pseudohydrostatic, are not considered in the layout of the solution scheme and they must be verified only with reference to the final configuration of the resistant truss. However, even if a wide literature and special publications present solutions for many cases of the practice [6], the problem in creating a S&T model of an arbitrary given structure remains open. As a consequence the actual shape of the truss depends on the intuition and on the experience of the designer, who, in any case, must verify that its model can be developed effectively, even though from a theoretical point of view it is conservative. For these reasons, the formulation of methods able to individuate a proper S&T model in a systematic way, results of great importance.

In this work a procedure for the automatic search for optimal S&T models in R.C. elements is proposed. The bases of the method and the criteria adopted to make the model work according to the behaviour of the given continuum have been shown in previous papers [3, 4, 5], and are here recalled. The procedure is developed for 2D systems but can be extended to 3D cases. The pratical application of the method is shown by giving some examples which lead us to appreciate the capacities of the optimal models in handling complex stress paths through schemes which are rational and suitable for a safety and consistent design.

## 2. Optimization of S&T models

We consider a generic structural element with a given geometry and assigned boundary conditions (Fig. 1a). The region within this element defines the existence domain of the admissible S&T models. Given that in Nature the load transmission happens in such a way that the associated strain energy results minimum, a rational design philosophy will search for the maximum stiffness truss [15].

A search for absolute optima requires a selection from an infinite set of possible trusses. An approximation of the optimum can be achieved by covering the assigned continuum domain with a closely spaced grid of $n$ nodal points interconnected by $m \leq n(n-1)/2$ bar elements and assuming this network as the new existance domain [2, 24]. Clearly the net of the nodes of this *basic truss* must include all the load points and all the supports. Therefore eventual distributed loads and continuous supports will be respectively represented by statically equivalent concentrated loads and by consistent punctual restraints (Fig. 1b).

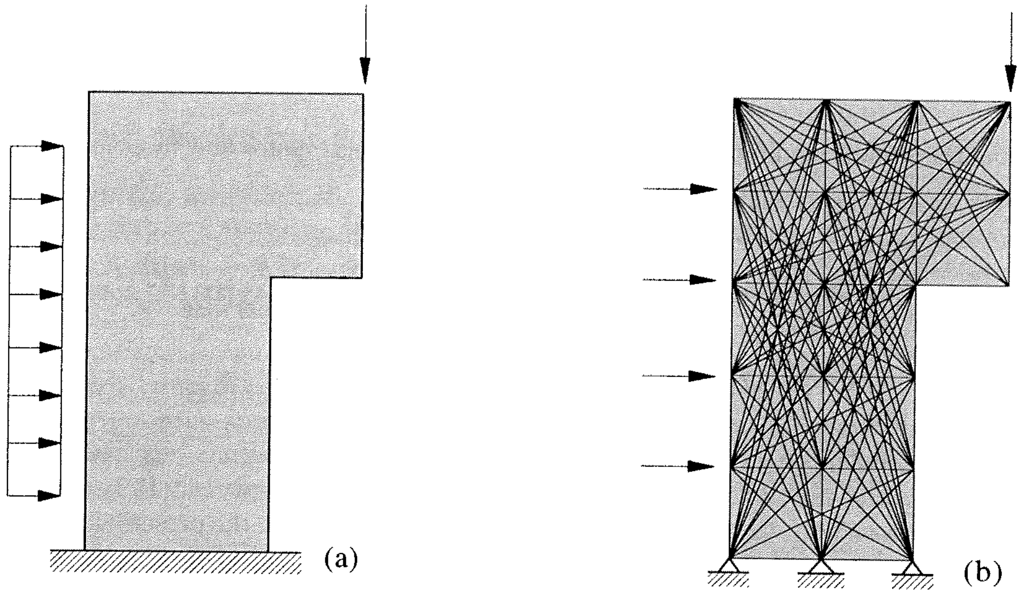


**Fig. 1.** (a) The structural element with assigned loads and displacements; (b) Basic truss

## 2.1. Equilibrium and conformity equations

With reference to the symbols in Fig. 2, we can write the equilibrium equation of the generic bar $k$, in the local $(x', y')$ and in the global $(x, y)$ reference systems, rotated, with respect to each other, by the angle $\beta_k$ [16]:

$$\begin{bmatrix} f_{x,i}^{k\,\prime} \\ f_{y,i}^{k\,\prime} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} n_k \qquad \begin{bmatrix} f_{x,j}^{k\,\prime} \\ f_{y,j}^{k\,\prime} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} n_k \qquad \Rightarrow \qquad \begin{array}{l} \mathbf{f}_i^{k\prime} = \mathbf{h}_i^{k\prime} n_k \\ \mathbf{f}_j^{k\prime} = \mathbf{h}_j^{k\prime} n_k \end{array}$$

$$\mathbf{T}_k = \begin{bmatrix} \cos\beta_k & -\sin\beta_k \\ \sin\beta_k & \cos\beta_k \end{bmatrix} \qquad \Rightarrow \qquad \begin{array}{l} \mathbf{f}_i^k = \mathbf{T}_k \mathbf{f}_i^{k\prime} = \mathbf{T}_k \mathbf{h}_i^{k\prime} n_k = \mathbf{h}_i^k n_k \, , \\ \mathbf{f}_j^k = \mathbf{T}_k \mathbf{f}_j^{k\prime} = \mathbf{T}_k \mathbf{h}_j^{k\prime} n_k = \mathbf{h}_j^k n_k \, . \end{array}$$

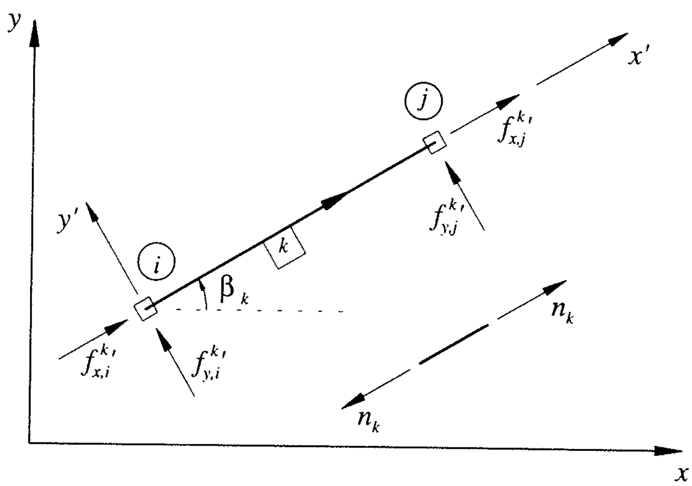


**Fig. 2.** Reference systems and sign conventions

By assembling the force vectors converging to a generic node $s$:

$$\mathbf{f}_s = \sum_{k \to s} \mathbf{f}_s^k$$

one obtains the overall equilibrium equation for a truss having $n$ nodes and $m$ elements:

$$\begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \cdots \\ \mathbf{f}_n \end{bmatrix} = \begin{bmatrix} \mathbf{h}_1^1 & \mathbf{h}_1^2 & \cdots & \cdots & \mathbf{h}_1^m \\ \mathbf{h}_2^1 & \mathbf{h}_2^2 & \cdots & \cdots & \mathbf{h}_2^m \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{h}_n^1 & \mathbf{h}_n^2 & \cdots & \cdots & \mathbf{h}_n^m \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ \cdots \\ \cdots \\ n_m \end{bmatrix}, \qquad \mathbf{h}_s^k = \begin{cases} \mathbf{h}_i^k, & \text{if } k \to s \text{ with } i \\ \mathbf{h}_j^k, & \text{if } k \to s \text{ with } j \\ \mathbf{0}, & \text{otherwise} \end{cases}$$

or:

$$\mathbf{f} = \mathbf{H}\mathbf{r}$$

where $\mathbf{f}$ is the vector of nodal forces, $\mathbf{r}$ is the vector of the axial forces and $\mathbf{H}$ is the equilibrium matrix. After the assembly, the previous system will be modified to take the prescribed displacements through the known methods of the structural analysis into account. In the following we will implicitly assume that the equilibrium matrix $\mathbf{H}$ has rank $2n < m$. This assumption justifies the search for an optimal solution. Finally, the vector $\mathbf{r}$ of the axial forces in the bars must comply with the conformity conditions:

$$-\mathbf{r}^- \leq \mathbf{r} \leq \mathbf{r}^+$$

where $\mathbf{r}^- \geq \mathbf{0}$ and $\mathbf{r}^+ \geq \mathbf{0}$ are respectively the limits due to the tension and compression strength of the $m$ elements.

## 2.2. Maximum stiffness–minimum volume truss

For a given load condition, the truss of maximum stiffness coincides with that of the minimum volume of material [10]. If one calls respectively $a_k$ and $l_k$ the area of the cross section and the length of the generic bar $k$, the total volume of the system results:

$$V = \sum_{k=1}^{m} a_k l_k = \mathbf{l}^T \mathbf{a}\,.$$

The problem in finding the truss having maximum stiffness and satisfying equilibrium and conformity conditions can be reduced to the following linear programming problem [9]:

$$\min\left\{\mathbf{l}^T\mathbf{a} \mid \mathbf{H}\mathbf{r} = \mathbf{f},\ \mathbf{r} \leq \sigma^+\mathbf{a},\ \mathbf{r} \geq -\sigma^-\mathbf{a},\ \mathbf{a} \geq \mathbf{0}\right\}$$

which involves $2n$ constraints and $2m$ variables $\mathbf{r}$ and $\mathbf{a}$. The stresses $\sigma^- \geq 0$ and $\sigma^+ \geq 0$ are respectively the level of tension and compression strength of the material. They are here assumed, for simplicity and without loss of generality, the same for all the elements.

If one introduces $2m$ additional variables $\mathbf{a}^-$ and $\mathbf{a}^+$, defined as follows:

$$\mathbf{r} - \sigma^+\mathbf{a} + (\sigma^- + \sigma^+)\,\mathbf{a}^- = \mathbf{0}$$
$$\mathbf{r} + \sigma^-\mathbf{a} - (\sigma^- + \sigma^+)\,\mathbf{a}^+ = \mathbf{0}$$

and removes the vector $\mathbf{r}$, the previous linear program can be reduced to the following normal form:

$$\min\left\{\mathbf{l}^T\left(\mathbf{a}^- + \mathbf{a}^+\right) \mid \mathbf{H}\left(\sigma^+\mathbf{a}^+ - \sigma^-\mathbf{a}^-\right) = \mathbf{f},\ \mathbf{a}^- \geq \mathbf{0},\ \mathbf{a}^+ \geq \mathbf{0}\right\}$$

again with $2n$ constraints and $2m$ variables $\mathbf{a}^-$ and $\mathbf{a}^+$, which respectively represent the possible areas of the cross section of the ties and of the struts ($a_k^- a_k^+ = 0$). In particular:

(a) if $a_k^- > 0$ and $a_k^+ = 0$, the element $k$ is a strut having $a_k = a_k^-$ and $r_k = -\sigma^-$;

(b) if $a_k^- = 0$ and $a_k^+ > 0$, the element $k$ is a tie having $a_k = a_k^+$ and $r_k = \sigma^+$;

(c) if $a_k^- = a_k^+ = 0$, the element $k$ doesn't belong to the optimal truss.

For the equivalence of the search criterion between tensioned and compressed elements, initially the following bounds $\sigma^- = \sigma^+ = \sigma$ are assumed and the linear programming problem results:

$$\min\left\{\mathbf{1}^T\left(\mathbf{a}^- + \mathbf{a}^+\right) \mid \sigma\mathbf{H}\left(\mathbf{a}^+ - \mathbf{a}^-\right) = \mathbf{f},\ \mathbf{a}^- \geq \mathbf{0},\ \mathbf{a}^+ \geq \mathbf{0}\right\}.$$

When the optimum truss is found, the actual resistant sections of the bars are deduced on the basis of the actual limit strength of the materials $\sigma^-$ and $\sigma^+$.

In truth the previous approach ignores the actual two-dimensional behaviour of the assigned continuum system and the assumed one-dimensional model has to be considered as a qualitative reference. Hence for a consistent design some new criteria are needed.

### 2.3. Improvement of the optimum criterion

The guidelines to improve the optimization process can be derived from a critical proof of the model just found. In particular, by starting from the same initial *basic truss*, a more refined force path can be deduced if some condition of similarity to the actual stress field in the given linear-elastic or non-linear-cracked continuum is imposed. To this aim, the following new bounds can be introduced:

(1) Each bar of the *basic truss* has to be enforced to work as a strut or a tie according to the type of the prevailing stress around its location.

(2) The objective function $V$ has to be optimized by giving a more relevant role (precisely a lower penalty weight) to the bars whose orientation results, on average, closer to one of the principal directions in their zone.

(3) For the tensioned elements, to be built through steel bars, a regular and straight tracing is preferable. Hence a lower penalty weight again has to be attributed to the longer ties instead of to the shorter ones. However, curved trajectories can be better fitted by short elements. Hence by giving a lower penalty weight to the shorter struts, one increases the capability to better adapt the compressed members to the local stress path.

All these conditions are now translated into mathematical bounds to search for an improved optimum truss. We suppose that the same two-dimensional element has been studied separately as a 2D continuum. We call it *reference continuum*.

Let $\sigma_1(x,y)$, $\sigma_2(x,y) \leq \sigma_1(x,y)$ and $0 \leq \alpha(x,y) \leq \pi/2$ respectively be the principal stresses and the angle which the direction 1 forms with the axis $x$. Hence, along each element $k$ $\sigma_1(x')$, $\sigma_2(x')$ and $\alpha(x')$, are known functions as well as the angles $\gamma_{1k}(x') \leq \pi/2$ and $\gamma_{2k}(x') = \pi/2 - \gamma_{1k}(x')$ shown in Fig. 3. By assuming:

$$p = \left\{\, t \mid \gamma_{tk}(x') = \min\left[\gamma_{1k}(x');\ \gamma_{2k}(x')\right] \leq \pi/4,\ t = 1,2\right\},$$
$$q = \left\{\, t \mid \gamma_{tk}(x') = \max\left[\gamma_{1k}(x');\ \gamma_{2k}(x')\right] \geq \pi/4,\ t = 1,2\right\},$$

we can define an average angular deviation of the bar $k$ with respect to trajectories of the principal directions which intersect it:

$$\gamma_k = \frac{1}{l_k}\int_0^{l_k} \gamma_{pk}(x')\mathrm{d}x'.$$

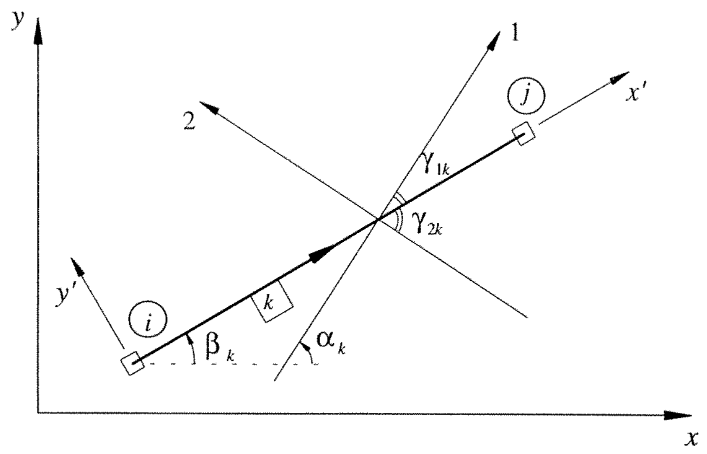


**Fig. 3.** Orientation of the generic bar in the principal stress field

We can again define the following parameters:

$$\sigma_{\|k} = \frac{1}{l_k}\int_0^{l_k} \sigma_{pk}(x')\cos\gamma_{pk}(x')\mathrm{d}x', \qquad \sigma_{\perp k} = \frac{1}{l_k}\int_0^{l_k} \sigma_{qk}(x')\sin\gamma_{qk}(x')\mathrm{d}x',$$

which in average indicate the type of the prevailing stress around the element. The parameter $\sigma_{\|k}$ leads us to recognise if a bar is a strut or a tie. The number of variables, in this way, halves and the linear programming problem assumes the following form:

$$\min\left\{\mathbf{l}^T\mathbf{a} \mid \sigma\bar{\mathbf{H}}\mathbf{a} = \mathbf{f},\ \mathbf{a} \geq \mathbf{0}\right\}$$

where:

$$\bar{\mathbf{H}} = \begin{bmatrix} \bar{\mathbf{h}}_1^1 & \bar{\mathbf{h}}_1^2 & \dots & \dots & \bar{\mathbf{h}}_1^m \\ \bar{\mathbf{h}}_2^1 & \bar{\mathbf{h}}_2^2 & \dots & \dots & \bar{\mathbf{h}}_2^m \\ \dots & \dots & \dots & \dots & \dots \\ \bar{\mathbf{h}}_n^1 & \bar{\mathbf{h}}_n^2 & \dots & \dots & \bar{\mathbf{h}}_n^m \end{bmatrix}, \qquad \bar{\mathbf{h}}_s^k = \frac{\sigma_{\|k}}{|\,\sigma_{\|k}\,|}\mathbf{h}_s^k\,.$$

Now, as mentioned, a closer modellization of the actual structural behaviour can be achieved by weighting the contribution of each bar according to its orientation with respect to the stress of the *reference continuum* and to its attitude in reproducing the continuum stress path. Let

$$\Gamma_k(\gamma_k) = \tan\gamma_k \in [0;1], \qquad \lambda_k(l_k) = \begin{cases} \dfrac{l_k}{\max\{l_k \mid k = 1,\dots,m\}} \in [0;1], & \text{if } \sigma_{\|k} > 0, \\ 1 - \dfrac{l_k}{\max\{l_k \mid k = 1,\dots,m\}} \in [0;1], & \text{if } \sigma_{\|k} \leq 0, \end{cases}$$

the weighting penalty function is chosen as follows:

$$w_k(\gamma_k, l_k) = \Gamma_k^{\mu\gamma_k} \in [0;1]$$

where $\mu \geq 0$ is a numerical coefficient heuristically deduced. With these assumptions, the new total volume is the sum of the single weighted volumes:

$$V_w = \sum_{k=1}^{m} w_k a_k l_k = (\mathbf{W}\mathbf{l})^T\mathbf{a} = \mathbf{l}_{eq}^T\mathbf{a}$$

where $\mathbf{W}$ is the diagonal matrix of the weights and $l_{eq}$ is the vector of the equivalent lengths. The final form of the linear programming problem is given by:

$$\min\left\{\mathbf{l}_{eq}^T\mathbf{a} \mid \sigma\bar{\mathbf{H}}\mathbf{a} = \mathbf{f},\ \mathbf{a} \geq \mathbf{0}\right\}.$$

Now the S&T models result consistent with the stress field in the *reference continuum* and suitable for the actual design. However also in this case, especially if the *basic truss* grid is not closely spaced, some lacks with respect to the design practice can appear. In particular, some elements may fall on the edges of the given dominion and many local stress diffusions around the points of application of the loads or around the supports may be completely ignored. Finer nets can certainly give better results, but they involve a definitely higher computational effort, which is a function of $n^2$. At this point, a different way for a more refined research of the S&T model uses the aforementioned criteria, but now applied to the new *basic truss* represented by the family of the already specialized truss given by the parametric transformation of the nodal coordinates which define the shape of the previously optimized truss. These developments are shown in the application section.

### 2.4. Dimensioning the bars

Let $f_{cd}$ the design uniaxial compression strength of the concrete. In the struts the design limit strength is assumed as [22]:

$$f_{cd}^* = \nu_c(0.85 f_{cd}) = \nu_c f_{c1d}$$

where the coefficient 0.85 takes the long term effects into account and $\nu_c$ is an *efficiency factor* which is assumed according to the actual stress state of the *reference continuum* as follows:

$$\nu_{ck} = \begin{cases} 0.2(4 - \Gamma_k) & , \text{ if } \sigma_{\perp k} > 0, \\ 1.0 & , \text{ if } \sigma_{\perp k} \leq 0. \end{cases}$$

In a similar way, we can write $f_{yd} = \nu_s f_{c1d}$ for the steel. By putting $\sigma = f_{c1d}$, it is then possible to evaluate the effective areas of the resistant section of the optimal S&T model:

$$\begin{cases} a_{sk} = a_k/\nu_{sk} & , \text{ if } \sigma_{\|k} > 0, \\ a_{ck} = a_k/\nu_{ck} & , \text{ if } \sigma_{\|k} \leq 0. \end{cases}$$

## 3. Applications

A general search strategy for an optimal S&T model can be summarized through the following steps, as shown in the corresponding windows of Figs. 5–9 in more detail.

(1) A F.E.A. gives the internal state of the given structural element and, in particular, its field of the principal stresses. Such an analysis may consider the structure as linear-elastic or cracked and non-linear in both the component materials [17]. From a theoretical point of view the S&T model, being related to the lower bound theorem, is independent with respect to kinematics and then may refers indifferently to the working or to the ultimate state. But, to this purpose, it must be remembered that, while for ideal rigid-plastic systems the lower bound theorem can be directly applied, its use in dealing with R.C. elements requires checks of their actual deformability [4]. So if the assumed load path differs sensibly with respect to those which develop in the cracked structure, it may not be considered among the actual possible redistribution schemes. For these reasons the design usually refers to the working states and, hence, to the stress field given by the linear elastic analysis.

(2) After the boundary conditions are discretized in such a way that the forces acting on one face of the structure will be self-equilibrated or in equilibrium with those acting on an opposite face, as suggested by the load path method, a first *basic truss* defined by a closely spaced grid of $n$ nodal points interconnected by $m$ bar elements is constructed within the region of the assigned continuum. In this domain we search for the maximum stiffness truss by attributing at first the same weight ($\mu$ is set to zero), and then selected weights ($\mu$ is set to unity) to the volumetric contribution of each bar in function of its affinity to the actual stress flow.

(3) Based on last model, with the aforementioned criteria ($\mu = 1$) a searches for a more refined S&T model in a new *basic truss* generated by moving its nodes position in a properly way. The forces acting on the bars of the S&T model so obtained are used to verify the concrete struts and to dimension the steel ties.

(4) The reliability of the internal forces path, which the optimal S&T model implicitly defines, is evaluated by a comparison with the direction of the principal stresses, respectively maximum and minimum, and with the stress diagram along significant sections.

The previous procedure is now applied in the analysis of a prismatic element of depth $d$ and subjected to localized loads, as in the case of the anchorage zone of a post-tensioned beam. The prismatic element is extracted from the whole structure according to the De Saint Venant principle. The main factors which influence the stress diffusion process are the number, the extension $a$ and the relative position of the loaded areas [8, 25]. In the following, localized loads having the same intensity $P$ and a loaded area/depth ratio $a/d = 0.1$ are considered. In particular, the load conditions shown in Fig. 4, obtained by varying the number and the arrangement of these loads in some basic combinations, are investigated.

Figs. 5–9 show, in the corresponding windows, all the phases of the definition of the optimal S&T models. The adimensional values of the axial forces and the corresponding efficiency factors are listed in Table 1. All the presented S&T models agree with the solutions presented in literature.

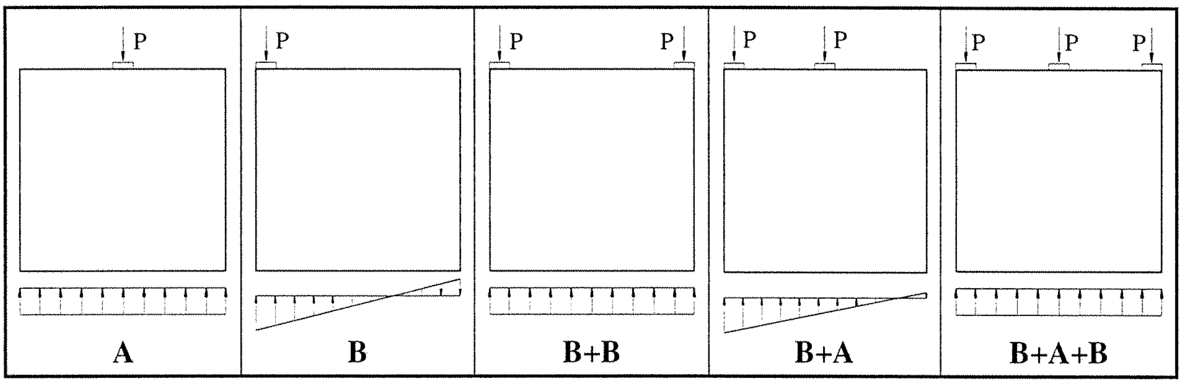


**Fig. 4.** Prismatic structural element subjected to several load conditions

**Table 1.** Optimal S&T model: member forces $n_k = a_k\sigma$ and efficiency factors $\nu_k$

| — | A | | B | | B + B | | B + A | | B + A + B | | — |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Bar | $n_k/P$ | $\nu_k$ | $n_k/P$ | $\nu_k$ | $n_k/P$ | $\nu_k$ | $n_k/P$ | $\nu_k$ | $n_k/P$ | $\nu_k$ | Bar |
| 1 | −0.500 | 1.000 | −1.000 | 1.000 | −1.000 | 1.000 | −1.000 | 1.000 | −1.000 | 1.000 | 1 |
| 2 | −0.553 | 0.757 | −1.026 | 1.000 | −1.061 | 0.790 | −1.000 | 1.000 | −0.500 | 0.739 | 2 |
| 3 | −0.500 | 0.782 | −0.353 | 0.763 | −1.000 | 1.000 | −1.011 | 1.000 | −1.054 | 0.781 | 3 |
| 4 | −0.237 | 1.000 | −1.000 | 1.000 | −0.356 | 1.000 | −1.008 | 1.000 | −0.601 | 0.730 | 4 |
| 5 | 0.237 | $\nu_{s5}$ | −0.268 | 1.000 | 0.356 | $\nu_{s5}$ | −0.048 | 1.000 | −1.000 | 1.000 | 5 |
| 6 | — | — | −0.230 | 1.000 | — | — | −1.000 | 1.000 | −0.500 | 1.000 | 6 |
| 7 | — | — | 0.230 | $\nu_{s7}$ | — | — | −1.000 | 1.000 | −0.333 | 1.000 | 7 |
| 8 | — | — | 0.268 | $\nu_{s8}$ | — | — | −0.045 | 1.000 | 0.333 | $\nu_{s8}$ | 8 |
| 9 | — | — | — | — | — | — | −0.145 | 1.000 | 0.333 | $\nu_{s9}$ | 9 |
| 10 | — | — | — | — | — | — | −0.017 | 1.000 | — | — | 10 |
| 11 | — | — | — | — | — | — | 0.145 | $\nu_{s11}$ | — | — | 11 |
| 12 | — | — | — | — | — | — | 0.017 | $\nu_{s12}$ | — | — | 12 |
| 13 | — | — | — | — | — | — | 0.045 | $\nu_{s13}$ | — | — | 13 |

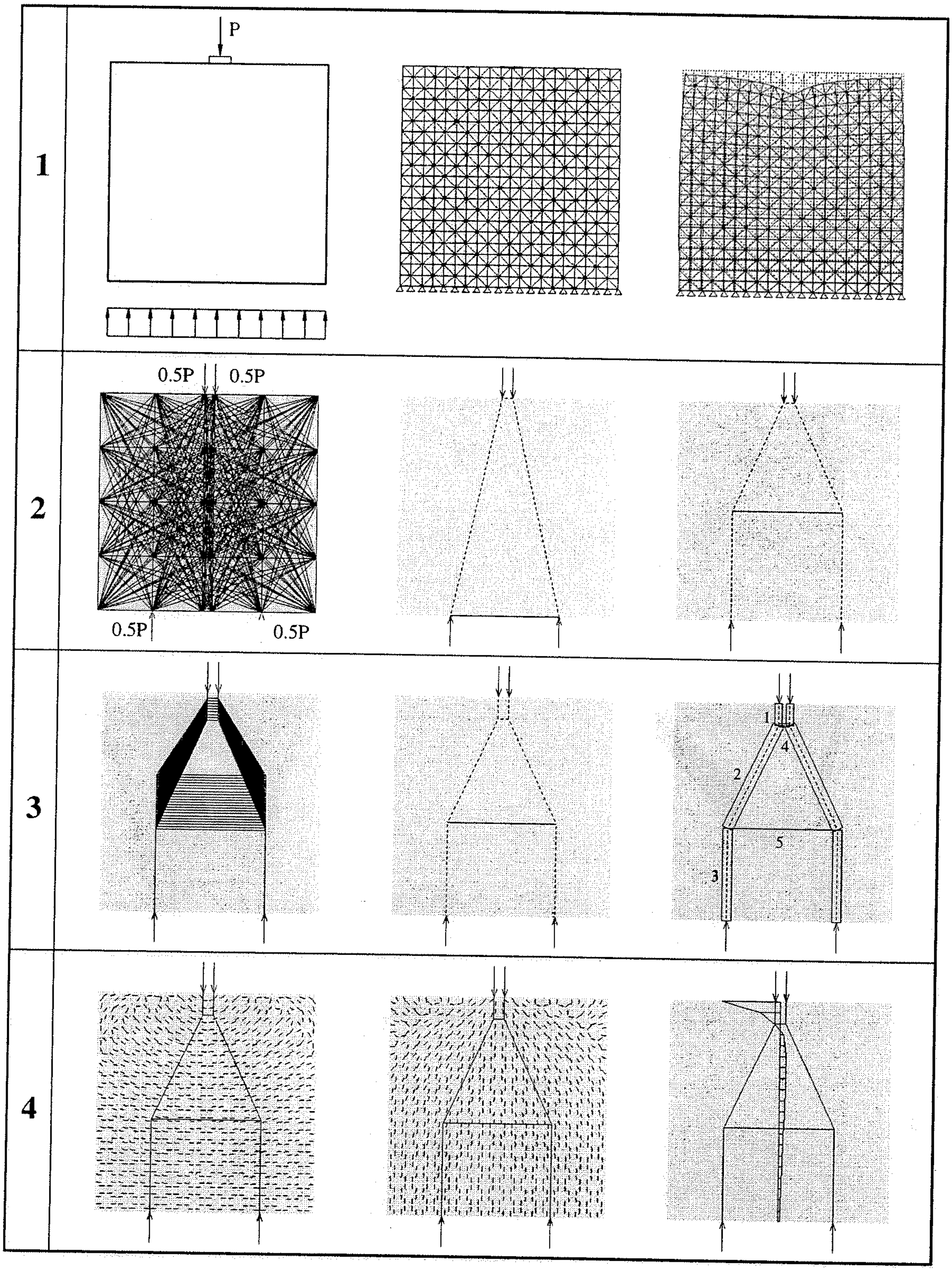


**Fig. 5.** Search for an optimal S&T model: case A

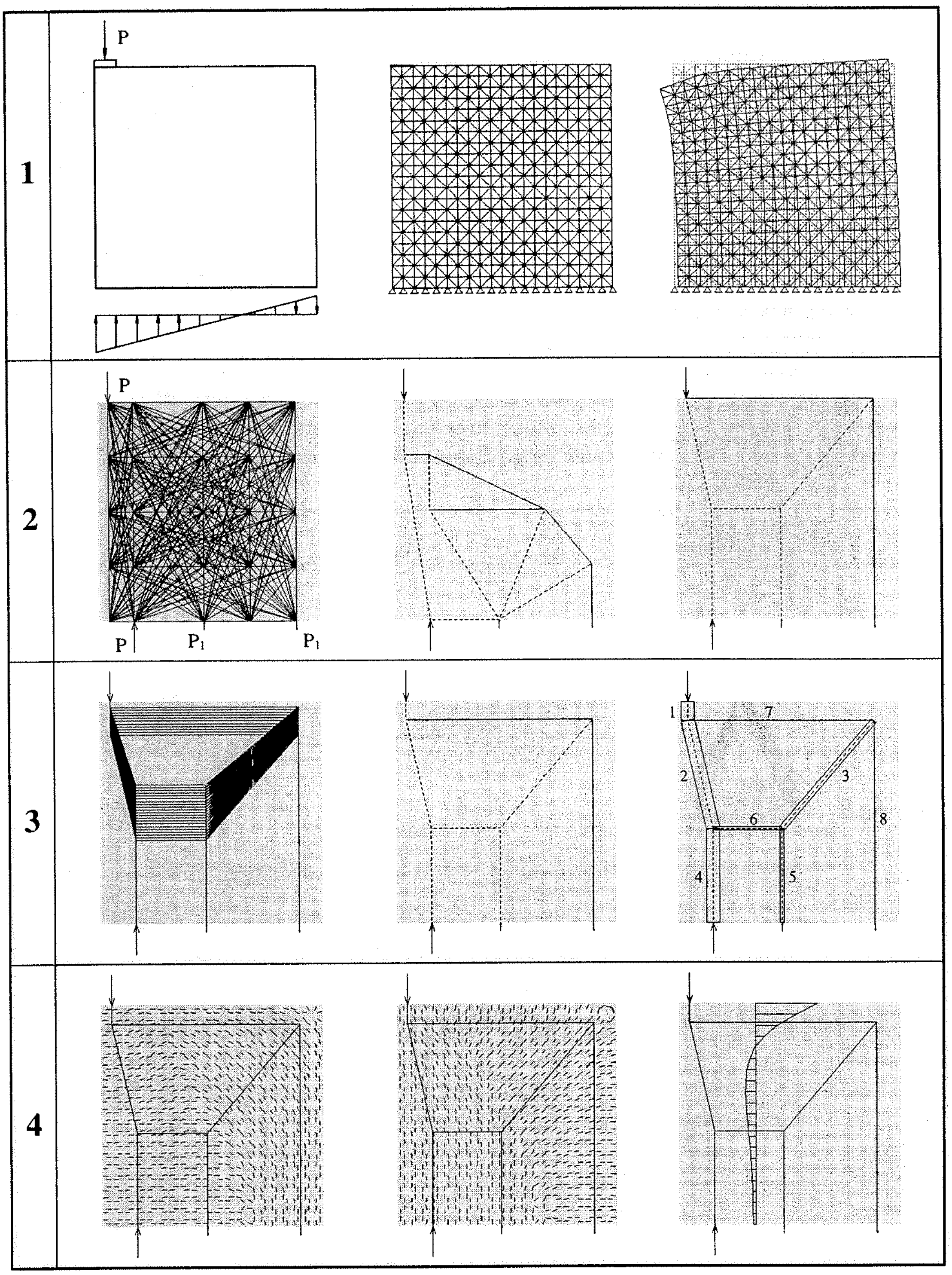


**Fig. 6.** Search for an optimal S&T model: case B

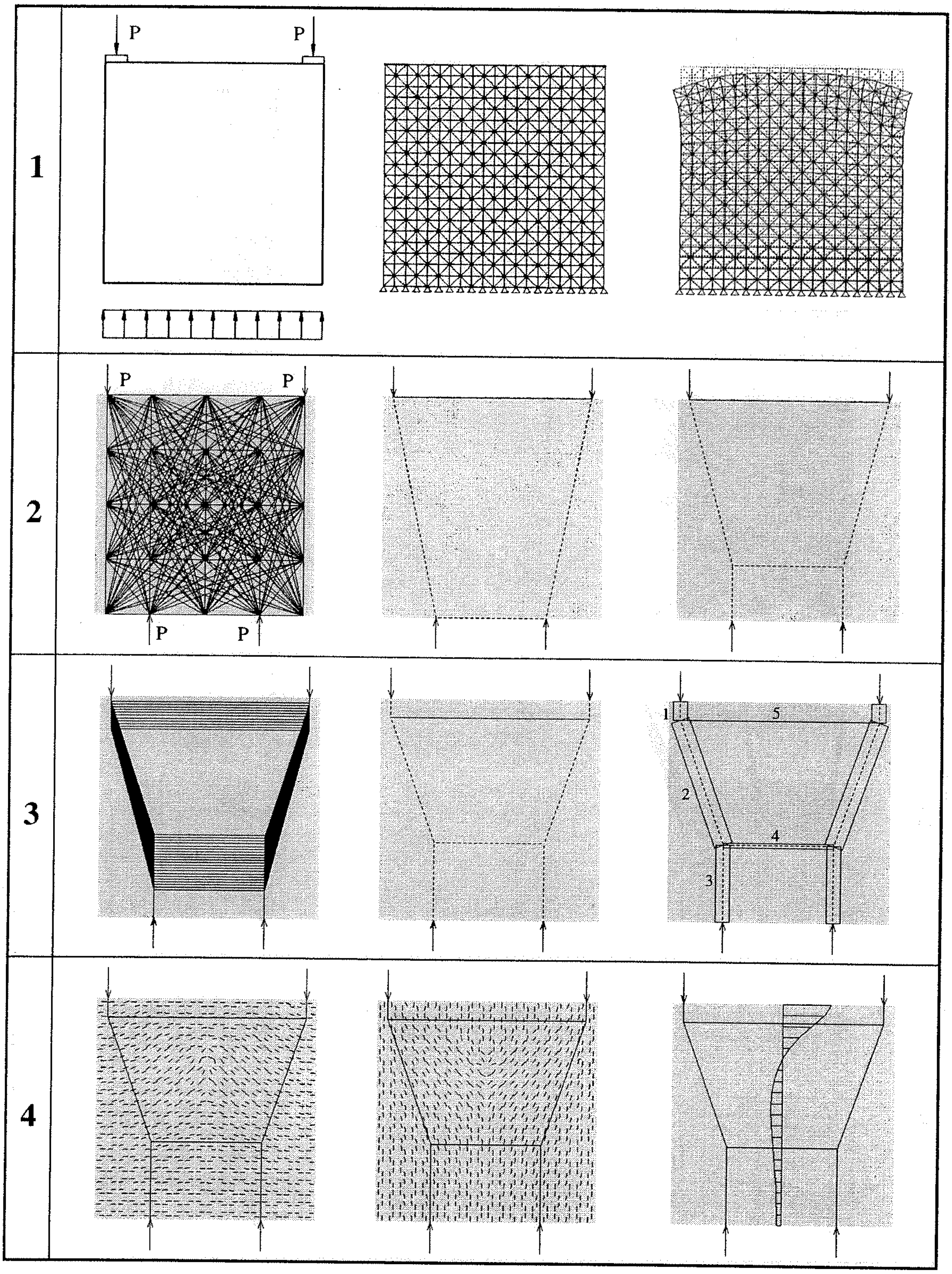


**Fig. 7.** Search for an optimal S&T model: case B+B

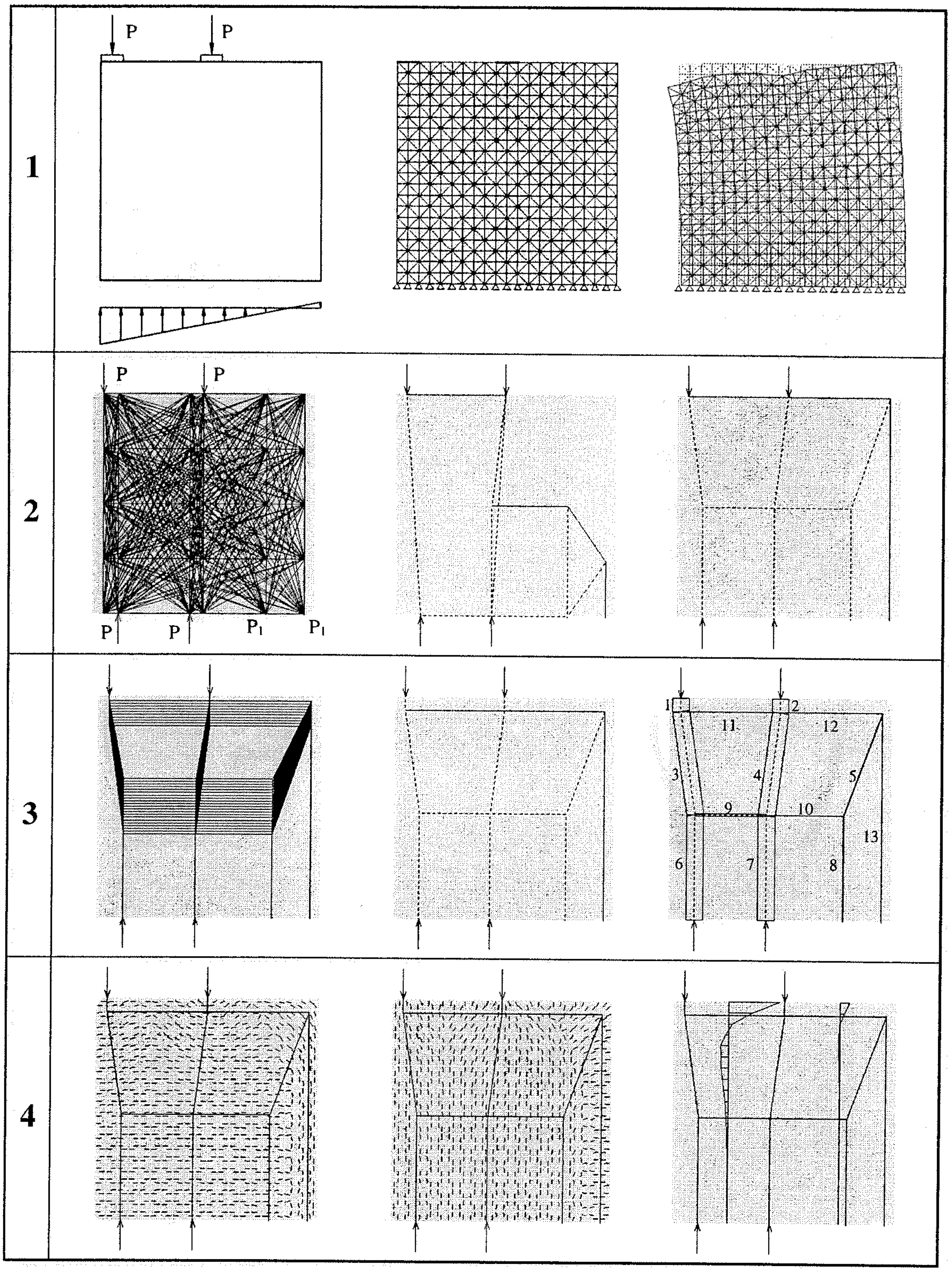


**Fig. 8.** Search for an optimal S&T model: case B+A

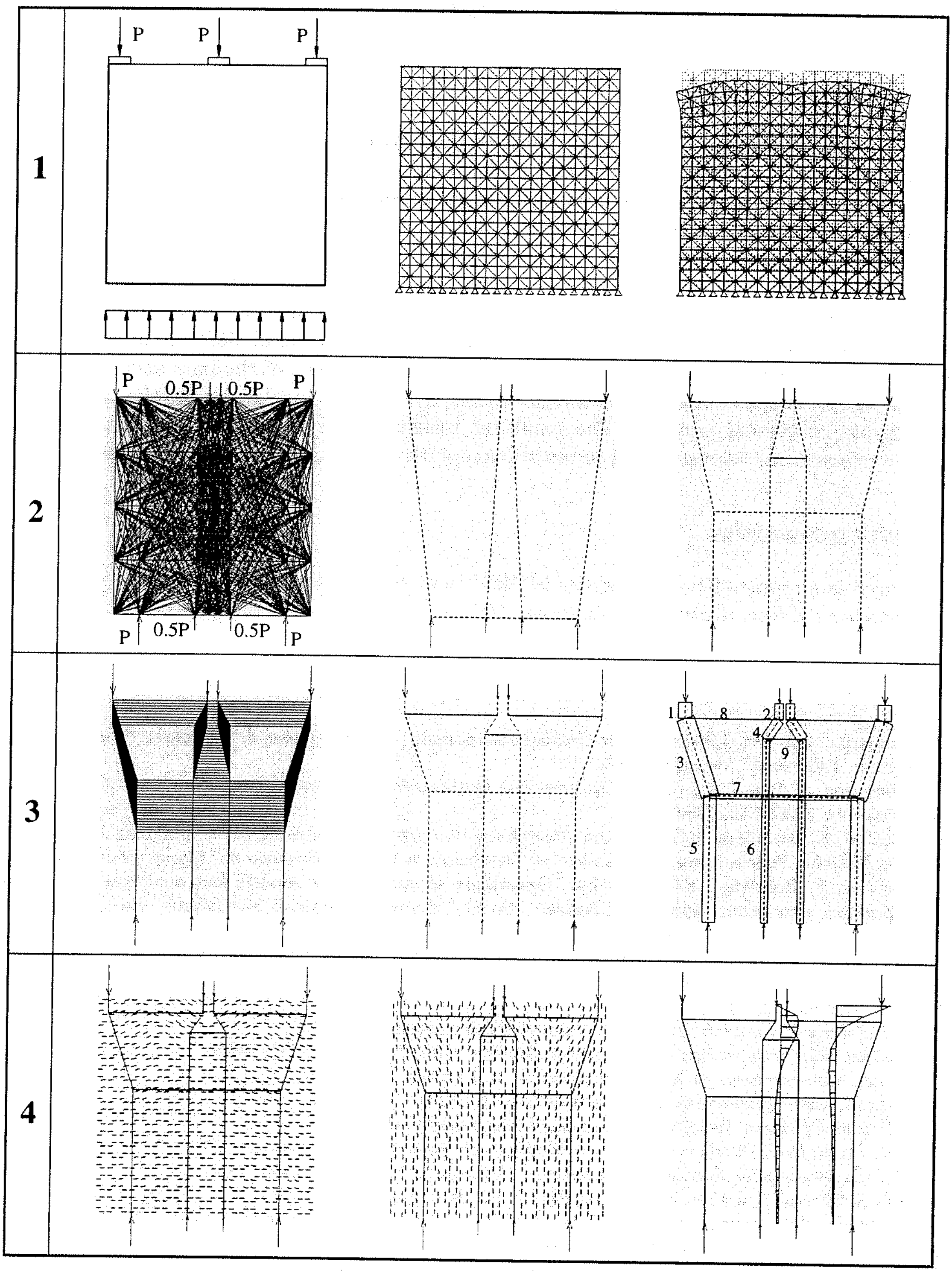


**Fig. 9.** Search for an optimal S&T model: case B+A+B

In particular, one can observe that the alignment of the bars accurately averages the flow of the actual stresses and that the principal struts and ties are localized near the centroid of the stress diagram of typical sections. The resultant tension and compression forces, as well as the relative lever arms, are accurate and in accordance with the theory and the experiments.

## 4. CONCLUSIONS

A procedure for the automatic search for optimal S&T models in R.C. elements is proposed. The structural problem is discretized by replacing the assigned continuum domain by a suitable *basic truss* and is translated into a mathematical linear programming problem. In particular, the layout optimization process is carried out by weighting the contribution of each bar to the objective function (the volume of the truss) according to a given reference stress field.

A comparison between the models obtained disregarding the actual two-dimensional behaviour ($\mu = 0$) and the optimal one ($\mu = 1$) lead us to appreciate the capacities of the proposed optimum criterion in handling complex stress paths. In particular, the optimal S&T models show a good agreement with the solutions presented in literature. The alignment of the bars accurately averages the flow of the actual stresses. The principal struts and ties are localized near the centroid of the stress diagram of typical sections. The resultant tension and compression forces, as well as the relative lever arms, are accurate and in accordance with the theory and the experiments.

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## REFERENCES

[1] P. Baumann. *Die Druckfelder bei der Stahlbetonbemessung mit Stabwerkmodellen.* Dissertation. Institut für Massivbau, Universität Stuttgart, 1988.

[2] M.P. Bendsøe, A. Ben-Tal, J. Zowe. Optimization methods for truss geometry and topology design. *Structural Optimization*, **7**:141–159, 1994.

[3] F. Biondini, F. Bontempi, P.G. Malerba. Search for strut-and-tie models by linear programming (in Italian). *Studi e Ricerche*, Scuola di Specializzazione in Costruzioni in C.A., Politecnico di Milano, **17**:121–156, 1996.

[4] F. Biondini, F. Bontempi, P.G. Malerba. Generation of strut-and-tie models and nonlinear analysis of the corresponding structural response (in Italian). *Studi e Ricerche*, Scuola di Specializzazione in Costruzioni in C.A., Politecnico di Milano, **18**:31–56, 1997.

[5] F. Biondini, F. Bontempi, P.G. Malerba. Optimization of strut-and-tie models in reinforced concrete structures. In: G.P. Steven, O.M. Querin, H. Guan, Y.M. Xie, eds., *Structural Optimization*, 115–122. Oxbridge, Sydney, 1998.

[6] CEB. *Detailing of Concrete Structures.* Bulletin d'Information **150**, 1982.

[7] W.F. Chen. *Plasticity in Reiforced Concrete.* McGraw-Hill, New York (NY), 1982.

[8] Y. Guyon. *Constructions en Béton Précontraint.* **2**, Eyrolles, Paris, 1968.

[9] G. Hadley. *Linear Programming.* Addison–Wesley, Reading (MA), 1975.

[10] W.S. Hemp. *Optimum Structures.* Clarendon Press, Oxford, 1973.

[11] IABSE. *Plasticity in Reinforced Concrete.* Colloquium Report **29**, Copenhagen, 1979.

[12] IABSE. *Computational Mechanics of Concrete.* Colloquium Report **54**, Delft, 1987.

[13] IABSE. *Structural Concrete.* Colloquium Report **62**, Stuttgart, 1991.

[14] M. Jennewein. *Zum Bemessen des Stahlbetons mit Stabwerkmodellen.* Dissertation. Institut für Tragwerksentwurf und Konstruktion, Universität Stuttgart, 1989.

[15] P. Kumar. Optimal force transmission in reinforced concrete deep beams. *Computers and Structures*, **8**(2):223–229, 1978.

[16] R.K. Livesley. *Matrix Methods of Structural Analysis.* Pergamon Press, New York, 1975.

[17] P.G. Malerba, ed. *Limit and Nonlinear Analysis of Reinforced Concrete Structures* (in Italian). CISM, Udine, 1998.

[18] P. Marti. Basic tools of reinforced concrete beam design. *ACI Structural Journal*, **82**(1):46–56, 1985.